Bounding the Capacity of Saturation Recording: The Lorentz Model and Applications

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Abstract—**This paper concerns the problem of bounding the information capacity of saturation recording. The bounds apply to a variety of systems such as magnetic tape recorders, magnetic disks, and optical data disks. The superposition channel with additive Gaussian noise is used as a model for recording. This model says that for a *saturation* input signal, $x(t)$, (i.e., one that can assume only one of two levels) the output can be expressed as $y(t) = \tilde{x}(t) + z(t)$ where $\tilde{x}(t)$ is a filtered version of the input $x(t)$ and $z(t)$ is additive Gaussian noise. The channel is described by the impulse response of the channel filter, $h(t)$, and by the autocorrelation function of the noise. A specific example of such a channel is the differentiated Lorentz channel; this model is often used to describe magnetic recording [2]–[6]. Certain autocorrelation and spectrum expressions for a general Lorentz channel are derived. Upper and lower bounds on the capacity of saturation recording channels are described. The bounds are explicitly computed for the differentiated Lorentz channel model. Finally, it is indicated how the derived bounds can be applied in practice using physical measurements from a recording channel.**

I. Introduction

THIS paper concerns the question of determining the capacity of saturation systems to record information. The capacity of a recording device, such as a magnetic tape recorder or an optical data disk, is a measure of the maximum number of bits of information that can be reliably represented along each length of the track. The capacity is a number, measured in units of information per unit length (for example, bits per inch). To be a reliable representation of information requires that the data can be recovered with a small probability of error. The theory of information, as first described by Shannon [9], is a proven method of estimating the capacity of communications systems such as telephone line transmission and satellite communications. The method has proven itself to be an indispensable tool of digital communications engineering. This paper attempts to provide some similar tools useful for the analysis and design of digital recorders.

This paper derives formulas that provide upper and lower bounds on estimates of the capacity of saturation recording systems. These bounds are based on the new result in [1] and the classical techniques described in [8, ch. 7, 8, 19]. With these formulas, physical measurements obtained from a recording system can be used to obtain bounds on the capacity measured in bits per inch. These bounds can be used in the selection of individual physical compounds of the system (e.g., heads, media, electronics) and in the design of the signal processing elements (e.g., modulators, read and write equalizers, detectors). The bounds can also be used to compare the estimated capacity to the actual achieved information density of an existing recording system.

In order to obtain estimates on the capacity of a channel (transmission or storage), a simple model must be available that provides a reasonable approximation to the transformation of input signals to output signals. Models for communications channels often involve distortion on the input signal and noise (i.e., a deterministic and a random component). A typical model for a telephone or satellite channel is one for which there is linear filtering on the input and additive noise. Similar models for saturation recording channels have often been proposed and studied [2]–[6]. The main features that distinguish these two models are the constraints on the input and the form of the channel filtering.

In most communications channels, a constraint on the input signal $x(t)$ takes the form of a limit on the average or peak power of the input signal. For example, an average power constraint P would require that:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t))^2\, dt \le P$$

while a peak power constraint on the input would require $(x(t))^2 \le P$ for all t or, equivalently,

$$\frac{1}{T}\int_\tau^{\tau+T} (x(t))^2\, dt \le P$$

for all T, τ. In saturation recording, such as a magnetic or optical recording channel, a more severe constraint is imposed, namely $(x(t))^2 = P$ for all t. One might describe such a constraint as a "constant power" constraint since it is required that:

$$\frac{1}{T}\int_\tau^{\tau+T} (x(t))^2\, dt = P$$

for all T, τ.

In this paper, the techniques needed to obtain upper and

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lower bounds on the capacity are described. The upper bound uses the fact that the average power constraint is a less severe requirement and the capacity for this constraint is easily computed. The technique for determining the capacity under an average power constraint is commonly called "water filling" [8]. The lower bound follows from a new result, presented as a lemma, in the paper by Ozarow, Wyner, and Ziv [1]. This result allows one to obtain a lower bound on the capacity of a saturation channel as a lower bound to the peak power channel.

The superposition channel with additive Gaussian noise is used as a model for saturation recording. This model says that for the saturation input signal $x(t)$, the output can be expressed as $y(t) = \tilde{x}(t) + z(t)$ where $\tilde{x}(t)$ is a filtered version of the input $x(t)$ and $z(t)$ is additive Gaussian noise. The channel is described by the impulse response of the channel filter, $h(t)$, and by the autocorrelation function of the noise. A specific example of such a channel is the differentiated Lorentz channel; this model is often used to describe magnetic recording [2]–[6].

Upper and lower bounds on the capacity of saturation recording channels modeled by the superposition channel with saturation inputs are described. The bounds are explicitly computed for the differentiated Lorentz channel model.

The purpose of this paper is twofold.

The first purpose is to describe digital recording as a digital communications problem and to bridge the magnetic recording and communications/information theory areas. Basic results and relationships from information theory and simple models for the recording process are brought to bear on the problem of determining the capacity of digital recording systems. An attempt has been made to present a self-contained introduction to the techniques and models useful in dealing with these problems. It is the hope that the tools presented will spark an interest from those involved in building state-of-the-art digital recording systems and from those from the digital communications area.

Second, the expressions derived can be used to bound the capacity of recording systems modeled by the Lorentz response and from physical measurements from a particular recording channel. These bounds can be used as a guide in determining the performance of individual components as well as entire recording systems.

The paper goes as follows. First, the basic definitions of the channel are given. Through the notion of a matched filter, the waveform channel model is equated to a model with vector inputs and outputs. This is followed by a brief review of Fourier transforms and several related identities. A general Lorentz model for recording is presented and basic autocorrelation and spectrum results are derived.

Upper and lower bounds on the capacity of saturation recording channels are developed. First, bounds on the mutual information for the vector model are obtained. Next, the channel capacity theorem is described in terms of maximizing the average mutual information between the input and output. An extremely useful lemma, Lemma 1 of [1], is reviewed and its application to the saturation recording channel capacity question is explained. Combining the bounds on the mutual information with the capacity results, upper and lower bounds on the capacity of saturation recording are obtained.

As an example of the utility of these bounds, the bounds on the capacity are explicitly computed for the differentiated Lorentz channel model. Also, it is indicated how the derived bounds can be applied in practice from physical measurements on a saturation recording channel.

Finally, we note that recent progress has been made on improving the bounds that are employed here. In [15], improvements in the lower bound are described, while [16] shows some methods for improving the upper bound. These results fit well into the models we develop here and might prove useful in developing tools for bounding the capacity of a digital recorder.

II. Channel Model and Definitions

A. The Saturation Recording Channel Model

Saturation recording channels are often described by a superposition model with additive noise (see Fig. 1). In direct recording systems, the *input signal* $x(t)$ is chosen to saturate the channel; the signal assumes only one of two possible values at time t, $x(t) \in \{+\sqrt{P}, -\sqrt{P}\}$. In practice, the input signal is synchronized with a clock and thus can be expressed by:

$$x(t) = \sum_{j=1}^{n} x_j B_T(t - jT) \tag{1}$$

where T is the *symbol period*, the *symbols* $x_j \in \{+\sqrt{P}, -\sqrt{P}\}$, and the *box function*

$$B_T(t) = \begin{cases} 1, & 0 \le t < T; \\ 0, & \text{otherwise.} \end{cases}$$

The additive noise, superposition model produces an *output signal*

$$y(t) = \tilde{x}(t) + z(t) \tag{2}$$

where $\tilde{x}(t)$ is a filtered version of the saturation signal $x(t)$ and $z(t)$ is additive white Gaussian noise. The channel is described in terms of the *impulse response* of the filter, $h(t)$, and the spectrum of the noise, $R_z(\tau) \equiv E_z(t + \tau)z(t)$. In particular, the filtering of the input signal is expressed as the convolution of $x(t)$ with $h(t)$

$$\tilde{x}(t) = h * x(t) \equiv \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)\, d\tau$$

$$= \sum_{j=1}^{n} x_j p_T(t - jT)$$

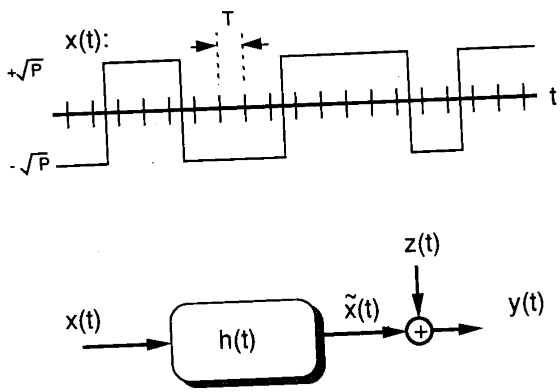

Fig. 1. Saturation signals and superposition channel.

where the *symbol pulse*, $p_T(t)$, is the convolution of the box function with the channel impulse response

$$p_T(t) = h * B_T(t) = \int_{-\infty}^{+\infty} h(\tau) B_T(t - \tau)\, d\tau$$
$$= g(t) - g(t - T)$$

where the channel *step response*

$$g(t) = \int_{-\infty}^{t} h(\tau)\, d\tau.$$

The *noise signal*, $z(t)$, is a zero-mean white Gaussian process that is independent of the input signal. The autocorrelation

$$R_z(\tau) = E\{z(t + \tau) z(t)\} = \frac{N_o}{2} \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function. [Note that the white noise assumption is not very confining; any noise that has a strictly positive power spectrum can be whitened by a linear filter. The whitening filter then becomes a part of the channel impulse response $h(t)$.]

It is important to note that the superposition model given by (2) holds for recording channels **only** when the input signal $x(t)$ is of the saturation type as in (1). For example, in magnetic recording, (2) does not describe the output signal $y(t)$ when the input $x(t)$ fails to saturate the magnetic material. This fact often leads to confusion when magnetic recording is described by a *linear* channel model. In addition, the model that is considered here is the so-called *direct recording* channel; it is not to be confused with the *AC-biased* channel which also has a *linear* channel description. (In AC-biased channels, the input signal is bounded but is not constrained to be of the saturation type.)

B. The Matched Filter Receiver

Given the output signal $y(t)$ [given by (2)], a sufficient statistic for the input signal $x(t)$ [given by (1)] is obtained by sampling the output of a filter matched to the symbol pulse $p_T(t)$ [18]. Let the matched filter output:

$$\tilde{y}(t) = p_{\bar{T}} * y(t) = \int_{-\infty}^{+\infty} p_T(-\tau) y(t - \tau)\, d\tau.$$

Then the samples

$$y_k \equiv \tilde{y}(kT), \qquad k = 1, 2, \cdots, n$$

form a sufficient statistic for $x(t)$ [i.e., these samples convey the same information about $x(t)$ that is contained in $y(t)$]. In this case, the matched filter has an impulse response that is equal to the time reverse of the symbol pulse $p_{\bar{T}}(t) \equiv p_T(-t)$. From this sufficient statistic, a vector model for the channel is obtained. Let the *input vector* $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^t$ and the *output vector* $\boldsymbol{y} = [y_1, y_2, \cdots, y_n]^t$. Then the two are related by the equation[1]

$$\boldsymbol{y} = M_T \boldsymbol{x} + \boldsymbol{z} \tag{3}$$

where the $n \times n$ *channel matrix* M_T is given by:

$$M_T = \{m_{ij}\}_{1 \le i,j \le n} \qquad m_{ij} = R_{p_T}((i - j)T) \tag{4a}$$

is determined by the *pulse autocorrelation function*

$$R_{p_t}(\tau) \equiv \int_{-\infty}^{+\infty} p_T(t + \tau) p_T(t)\, dt. \tag{4b}$$

The *noise vector* $\boldsymbol{z} = [z_1, z_2, \cdots, z_n]^t$ consists of samples of the noise signal, $z(t)$, at the output of the matched filter:

$$z_k = p_{\bar{T}} * z(kT) = \int_{-\infty}^{+\infty} p_T(-\tau) z(kT - \tau)\, d\tau.$$

This vector is a zero-mean Gaussian vector with autocorrelation matrix proportional to the channel matrix

$$E\{\boldsymbol{z}\boldsymbol{z}^t\} = \frac{N_o}{2} M_T$$

and is independent of the input vector $\boldsymbol{x}$.

In magnetic recording, a typical description of the channel impulse response, $h(t)$, is that of a lowpass filter and differentiator. One popular model for the lowpass filter is the *Lorentz pulse*, $1/(1 + t^2)$ [2], [6]. In this paper, upper and lower bounds on the Shannon information capacity of the superposition model for saturation recording channels are derived. These bounds are explicitly computed for the Lorentz channel response.

C. Fourier Transforms

In the sequel, the Fourier transform is used to describe many of the results. In particular, define the *Fourier*

[1]Note that the signal, $x(t)$, in (1) is only of saturation type for $0 \le t < nT$. To ensure that it is a saturation signal for all t means that x_k must be defined for both $k < 0$ and $k > n$ (e.g., set $x_k = +\sqrt{(P)}$, a constant). In this case, the matched filtered channel, $\boldsymbol{y} = M_T\boldsymbol{x} + \boldsymbol{z} + \boldsymbol{c}$, where the constant n-vector, $\boldsymbol{c}$, is known at both the input and output. Since the capacity in this case is independent of the value of $\boldsymbol{c}$, we assume without loss that $\boldsymbol{c} = 0$ from here on.

transform of a time signal $a(t)$ by the equation

$$A(f) = \int_{-\infty}^{+\infty} a(t) e^{i2\pi ft}\, dt.$$

Transform pairs and identities are then identified by the notation

$$a(t) \overset{\mathfrak{F}}{\leftrightarrow} A(f).$$

For example,[2]

$$B_T(t) \overset{\mathfrak{F}}{\leftrightarrow} \frac{\sin \pi fT}{\tau f} e^{i\pi fT}, \qquad \frac{1}{1+t^2} \overset{\mathfrak{F}}{\leftrightarrow} \pi e^{-2\pi|f|}, \quad \text{and}$$

$$\frac{t}{1+t^2} \overset{\mathfrak{F}}{\leftrightarrow} i\pi \operatorname{sgn}(f) e^{-2\pi|f|}.$$

Given a channel impulse response, $h(t)$, the Fourier transform

$$h(t) \overset{\mathfrak{F}}{\leftrightarrow} H(f)$$

is called the *channel transfer function*, $H(f)$. For the differentiated Lorentz channel,

$$h(t) = \frac{-2t}{(1+t^2)^2} \overset{\mathfrak{F}}{\leftrightarrow} H(f) = -i2\pi^2 f e^{-2\pi|f|}. \tag{5a}$$

Several identities are useful

$$a * b(t) \overset{\mathfrak{F}}{\leftrightarrow} A(f)B(f), \qquad \frac{d}{dt} a(t) \overset{\mathfrak{F}}{\leftrightarrow} i2\pi f A(f)$$

$$\hat{a}(t) \equiv \int_{-\infty}^{+\infty} \frac{a(\tau)}{t-\tau}\, d\tau \text{ (Hilbert Transform)}$$

$$\overset{\mathfrak{F}}{\leftrightarrow} i \operatorname{sgn}(f) A(f),$$

$$ca\left(\frac{t-\nu}{\alpha}\right) \overset{\mathfrak{F}}{\leftrightarrow} c|\alpha| A(\alpha f) e^{i2\pi f\nu}$$

and

$$R_a(t) = \int_{-\infty}^{+\infty} a(\sigma + t) a(\sigma)\, d\sigma \overset{\mathfrak{F}}{\leftrightarrow} |A(f)|^2.$$

As an example of the last identity, the differentiated Lorentz channel has a pulse autocorrelation (4b)

$$R_{p_t}(t) = \frac{4\pi}{4+t^2} - \frac{2\pi}{4+(t-T)^2} - \frac{2\pi}{4+(t+T)^2} \overset{\mathfrak{F}}{\leftrightarrow} 4\pi^2 \sin^2(\pi fT) e^{-4\pi|f|}.$$

Another useful tool involves the *Fourier series* obtained by sampling a signal $a(t)$. If $A(f)$ is the Fourier transform of $a(t)$, then by the *Poisson sum formula* [7]

$$\sum_{k=-\infty}^{+\infty} a(kT) e^{i2\pi fk} = \frac{1}{T} \sum_{j=-\infty}^{+\infty} A\left(\frac{f-j}{T}\right).$$

This is known as the *aliased spectrum* of $a(t)$. Note that the *pulse power spectral density*, $S_{p_T}(e^{i2\pi f})$, obtained from the sampled autocorrelation is a *power spectral density*; it is always a nonnegative number

$$S_{p_T}(e^{i2\pi f}) \equiv \sum_{k=-\infty}^{+\infty} R_{p_T}(kT) e^{i2\pi fk} = \frac{1}{T} \sum_{l=-\infty}^{+\infty} \left| P\left(\frac{f-l}{T}\right) \right|^2 \geq 0.$$

For example, in the differentiated Lorentz channel, the pulse power spectrum is obtained by sampling the autocorrelation

$$S_{p_T}(e^{i2\pi f}) = \frac{4\pi^2}{T \sinh\left(\frac{2\pi}{T}\right)} \sin^2(\pi f) \cosh\left(\frac{4\pi}{T}\left(f - \frac{1}{2}\right)\right),$$

$$0 \leq f \leq 1 \tag{5b}$$

where $\cosh(x) \equiv e^x + e^{-x}/2$ and $\sinh(x) \equiv e^x - e^{-x}/2$.

D. The General Lorentz Channel

The general Lorentz model for magnetic recording is based on two basic components related by the Hilbert transform. The *horizontal* and *vertical* components are given by:

$$g_h(t) = \frac{1}{1+t^2}, \qquad g_v(T) = -\hat{g}_h(t) = \frac{t}{1+t^2}.$$

The Lorentz model says that the step response of the system is a linear combination derived from the two components

$$g_L(t) = a g_h\left(\frac{t-\nu}{\alpha}\right) + b g_v\left(\frac{t-\sigma}{\beta}\right)$$

for fixed constants $a, b, \alpha > 0, \beta > 0, \nu, \sigma$. Fig. 2 shows the decomposition of such a model. The autocorrelation of the general Lorentz step-response is given by the expression:

$$\begin{aligned} R_{g_L}(\tau) = {} & \left(\frac{a^2\alpha\pi}{2}\right) g_n\left(\frac{\tau}{2\alpha}\right) + \left(\frac{b^2\beta\pi}{2}\right) g_h\left(\frac{\tau}{2\beta}\right) \\ & - \left(\frac{ab\alpha\beta}{\alpha+\beta}\right)\left[g_v\left(\frac{\tau-\nu+\sigma}{\alpha+\beta}\right) - g_v\left(\frac{\tau+\nu-\sigma}{\alpha+\beta}\right)\right] \\ & \overset{\mathfrak{F}}{\leftrightarrow} a^2\alpha^2\pi^2 e^{-4\pi\alpha|f|} + b^2\beta^2\pi^2 e^{-4\pi\beta|f|} \\ & + 2ab\alpha\beta\pi^2 \operatorname{sgn}(f) \sin(2\pi f(\nu - \sigma)) \cdot e^{-2\pi(\alpha+\beta)|f|} \end{aligned}$$

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[2] $\operatorname{sgn}(f) = \begin{cases} -1, & f < 0; \\ 0, & f = 0; \\ +1, & f \geq 0. \end{cases}$

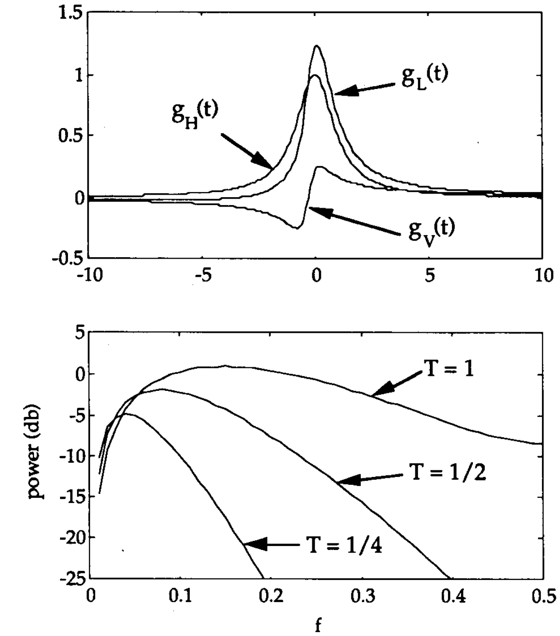


Fig. 2. General Lorentz pulse, $g_L(t)$ and $S_{p_T}(e^{i2\pi f})$ $a = 1$, $\alpha = 1$, $\nu = 0$, $b = 0.5$, $\beta = 0.5$, $\sigma = 0.3$).

while the pulse spectrum (aliased spectrum) for the pulse response, $p_T(t) = g_L(t) - G_L(t - T)$, is given by:

$$S_{p_T}(e^{i2\pi f}) = 4 \sin^2 (\pi f)$$

$$\cdot \Bigg[\frac{a^2\alpha^2\pi^2}{T \sinh (2\pi\alpha/T)} \cosh \left(4\pi\alpha\left(f - \frac{1}{2}\right)\Big/T\right) + \frac{b^2\beta^2\pi^2}{T \sinh (2\pi\beta/T)} \cosh \left(4\pi\beta\left(f - \frac{1}{2}\right)\Big/T\right)$$
$$+ \left(\frac{ab\alpha\beta\pi^2}{T \cosh (2\pi(\alpha + \beta)/T) - \cos (2\pi(\nu - \sigma)/T)}\right)$$
$$\cdot (\sin (2\pi(\nu - \sigma)f/T) \cosh (2\pi(\alpha + \beta)(1 - f)/T)$$
$$+ \sin (2\pi(\nu - \sigma)(1 - f)/T) \cosh (2\pi(\alpha + \beta)f/T)) \Bigg]$$

This last result can be derived via the identities presented in the Appendix.

III. Bounds on the Capacity

A. Mutual Information

The rate (or density) of reliable information storage in a saturation recorder is bounded by the information theoretic quantity known as the mutual information [8], [9]. For the channel of interest, given by (3):

$$y = M_T x + z. \tag{6a}$$

The input x and noise z are independent and the noise is zero-mean Gaussian with covariance proportional to M_T

$$x \perp z, z \sim \mathfrak{N}\left(0, \frac{N_o}{2} M_T\right). \tag{6b}$$

Since the input vector x represents stored information, it can be considered a random vector by assigning a distribution on the messages. In this case, the quantity that bounds the rate of reliable storage is

$$\frac{1}{nT} I(x; y) \quad \frac{\text{bits}}{\text{second}} = \frac{1}{n} I(x; y) \quad \frac{\text{bits}}{\text{symbol}}$$

where $I(x; y)$ is the *mutual information*. For the vector channel (6), the mutual information can be expressed:

$$I(x; y) = h(y) - h(y|x) = h(y) - h(z)$$

where the *differential entropy* of a random vector u

$$h(u) \equiv - \int f_U(u) \log (f_U(u)) \, du$$

is expressed in terms of the probability density of u, $f_U(u)$[3]. In the case of a Gaussian vector, such as z, this integral is easy to compute from the formula for differential entropy:

$$h(z) = \frac{n}{2} \log (\pi e N_o |M_T|^{1/n})$$

where $|M_T|$ is the determinant of the matrix M_T.[4]

In general, the differential entropy of the output, y, is difficult to compute. However, upper and lower bounds for $h(y)$ are well known.

The upper bound follows from the fact that the differential entropy of a random vector is bounded by the differential entropy of a Gaussian vector with the same covariance

$$h(y) \le \frac{n}{2} \log (2\pi e |R_y|^{1/n})$$
$$= \frac{n}{2} \log (\pi e |N_o M_T + 2 M_T R_x M_T^t|^{1/n})$$

where $R_y = E\{(y - Ey)(y - Ey)^t\}$ is the covariance of the output and $R_x = E\{(x - Ex)(x - Ex)^t|\}$ is the covariance of the input. In terms of the mutual information,

$$I(x; y) \le \frac{n}{2} \log \left(\left|I + \frac{2}{N_o} R_x M_T^t\right|^{1/n}\right).$$

[3] In this paper, all logarithms will be taken to the base 2; the information is measured in bits.

[4] Note that the determinant is nonnegative, $|M_T| \ge 0$, since M_T is a nonnegative definite matrix.

The average rate of data transmission, measured in bits per symbol, is obtained by considering the limit for large vector length n. Define the limiting average differential entropy:

$$h(X) \equiv \lim_{n\to\infty} \frac{1}{n} h(x),$$

$$h(Y) \equiv \lim_{n\to\infty} \frac{1}{n} h(y) \quad \text{and} \quad h(Z) \equiv \lim_{n\to\infty} \frac{1}{n} h(z).$$

Then the average mutual information

$$I(X; Y) \equiv \lim_{n\to\infty} \frac{1}{n} I(x, y) = h(Y) - h(Z) \qquad \frac{\text{bits}}{\text{symbol}}.$$

It is this average rate of information that plays an important role in determining the limits of reliable storage in saturation recording systems. Since the matrix M_T is obtained by sampling an autocorrelation function, it is a nonnegative definite, Toeplitz matrix. Thus, by using Szegö's theorem [10, pp. 64, 17], the limit

$$\lim_{n\to\infty} \frac{1}{n} \log (|M_T|) = \int_0^1 \log (S_{pT}(e^{i2\pi f}))\, df$$

is equal to the integral of the logarithm of the power spectral density, $S_{pT}(e^{i2\pi f})$ [e.g., (5b)]. This means that

$$h(Z) = \tfrac{1}{2} \int_0^1 \log (\pi e N_o S_{pT}(e^{i2\pi f}))\, df.$$

If the input vector, x, is obtained by sampling a stationary random process with power spectral density $S_x(e^{i2\pi f})$, then

$$h(Y) \le \tfrac{1}{2} \int_0^1 \log (\pi e S_{pT}(e^{i2\pi f})(N_o + 2S_x(e^{i2\pi f}) \cdot S_{pT}(e^{i2\pi f}))\, df.$$

(A careful proof of the validity of this bound involves the diagonalization of the channel matrix, M_T, in terms of its eigenvalues and eigenvectors. The problem involves the fact that the upper bound on $h(y)$ need not be of the form of a logarithm of the determinate of a *Toeplitz* matrix, a necessary condition for Szegö's theorem. However, if the covariance matrix, R_x, of the input vector, x, is carefully chosen, then the product $M_T R_x M_T^t$ will be Toeplitz for all n and the eigenvalues of R_x which converge to the given spectrum $S_x(e^{i2\pi f})$. In this case, the limit of the upper bound will be equal to the integral on the right. This procedure is analogous to the Karhunen–Loeve expansion for nonwhite noise [8, p. 398]).

This gives the following upper bound on the average mutual information

$$I(X; Y) \le \frac{1}{2} \int_0^1 \log \left(1 + \frac{2S_x(e^{i2\pi f}) S_{pT}(e^{i2\pi f})}{N_o}\right) df \quad \frac{\text{bits}}{\text{symbol}}. \tag{7}$$

The lower bound follows from the Entropy Power Inequality for vectors [9], [11], [12]

$$2^{(2/n)h(y)} \le 2^{(2/n)h(z)} + 2^{(2/n)h(M_T x)}$$

or

$$h(y) \ge \frac{n}{2} \log (\pi e N_o |M_T|^{1/n} + 2^{(2/n)h(M_T x)})$$

which holds for x independent of z. Using the identity

$$h(M_T x) = h(x) + \log (|M_T|)$$

give the bound

$$I(x; y) \le \frac{n}{2} \log \left(1 + \frac{2^{(2/n)h(x)} |M_T|^{1/n}}{\pi e N_o}\right)$$

on the mutual information. In the limit, the lower bound on the average mutual information

$$I(X; Y) \ge \frac{1}{2} \log \left(1 + \frac{2^{2h(X) + \int_0^1 \log (S_{pT}((e^{i2\pi f}))\, df}}{\pi e N_o}\right) \quad \frac{\text{bits}}{\text{symbol}}. \tag{8}$$

As an example of the bounds (7) and (8), consider an input vector x that is independent, identically distributed (i.i.d). If the vector is zero-mean Gaussian with variance P, $x \sim \mathfrak{N}(0, P)$, then

$$h(X) = \tfrac{1}{2} \log (2\pi e P), \qquad S_x(e^{i2\pi f}) = P$$

and

$$\frac{1}{2} \log \left(1 + \frac{2P}{N_o} 2^{\int_0^1 \log (S_{pT}(e^{i2\pi f}))\, df}\right) \le I(X; Y) \le \frac{1}{2} \int_0^1 \log \left(1 + \frac{2P}{N_o} S_{pT}(e^{i2\pi f})\right) df.$$

In the case that $S_{pT}(e^{i2\pi f}) = C$, a constant, the bounds meet and

$$I(X; Y) = \frac{1}{2} \log \left(1 + \frac{2PC}{N_o}\right).$$

Similarly, if the vector is uniformly distributed, $x \sim \mathfrak{U}[-\sqrt{P}, \sqrt{P}]$, then

$$h(X) = \log (2\sqrt{P}), \qquad S_x(e^{i2\pi f}) = \frac{P}{3}$$

and

$$\frac{1}{2} \log \left(1 + \frac{4P}{\pi e N_o} 2^{\int_0^1 \log (S_{pT}(e^{i2\pi f}))\, df}\right) \le I(X; Y) \le \frac{1}{2} \int_0^1 \log \left(1 + \frac{2P}{3N_o} S_{pT}(e^{i2\pi f})\right) df.$$

When $S_{pT}(e^{i2\pi f}) = C$, a constant, the bounds fail to meet and

$$\frac{1}{2}\log\left(1 + \frac{4PC}{\pi e N_o}\right) \le I(X; Y) \le \frac{1}{2}\log\left(1 + \frac{2PC}{3N_o}\right).$$

Note that the upper and lower bounds differ by a factor of $\pi e/6 \approx 1.423 \cdots \approx 1.53 \cdots$ (db) in signal-to-noise ratio.

B. Capacity

The information *capacity* determines the limit on the density at which reliable storage is possible. This limit is determined by maximizing the average mutual information normalized by the symbol period, $(1/T)\, I(X; Y)$ (bits/second). For the channel described by (4), the capacity, for a fixed period T, is the limit

$$C_T \equiv \overline{\lim_{n\to\infty}} \max_{p_n} \frac{1}{nT} I(x; y) \quad \frac{\text{bits}}{\text{second}} \tag{9a}$$

where $\mathfrak{P}_n$ is the set of allowable distributions on the input vectors x. If no constraint were placed on the input, then the capacity would be infinite. For saturation channels, the input vectors are constrained to lie on the vertices of an n-cube

$$\mathfrak{P}_n = \{f_X(x) | x \in \{-\sqrt{P}, +\sqrt{P}\}^n\} \tag{9b}$$

(i.e., each $x_j \in \{-\sqrt{P}, +\sqrt{P}\}$). The capacity of the channel is then determined by maximizing over the symbol period

$$C \equiv \sup_{T>0} C_T \quad \frac{\text{bits}}{\text{second}}.$$

It is not hard to argue that for any positive integer L, the inequality $C_T \le C_{T/L}$ holds. This implies that the capacity is obtained in the limit as the symbol period T goes to zero

$$C \equiv \overline{\lim_{T>0}}\, C_T \quad \frac{\text{bits}}{\text{second}}.$$

Determining the exact value of the capacity C_T or C for the constraint described by (9b) is a difficult open problem. However, a recently proven lemma provides a method for obtaining useful bounds on the capacity [1]. The present paper uses this lemma to obtain bounds on the capacity of saturation recording.

In pursuing upper and lower bounds, it is useful to consider two related measures of capacity. These notions of capacity are obtained by weakening the constraint imposed by the saturation channel (9b). The *peak power capacity* and *average power capacity* are given by:

$$C_T^p \equiv \overline{\lim_{n\to\infty}} \max_{\mathfrak{P}_n^p} \frac{1}{nT} I(x; y) \quad \frac{\text{bits}}{\text{second}},$$

$$C_T^a \equiv \overline{\lim_{n\to\infty}} \max_{\mathfrak{P}_n^a} \frac{1}{nT} I(x; y) \quad \frac{\text{bits}}{\text{second}}$$

$$C^p = \overline{\lim_{T\to 0}}\, C_T^p \quad \frac{\text{bits}}{\text{second}}$$

$$C^a = \overline{\lim_{T\to 0}}\, C_T^a \quad \frac{\text{bits}}{\text{second}}$$

where

$$\mathfrak{P}_n^p = \{f_X(x) | x \in [-\sqrt{P}, +\sqrt{P}]^n\},$$

$$\mathfrak{P}_n^a = \left\{f_X(x) | xx^t = \sum_{j=1}^{n} x_j^2 \le nP\right\}.$$

Note that each constraint can be described in terms of power. For the average, peak power and saturation constraint

$$\frac{1}{n}\sum_{j=1}^{n} x_j^2 \le P, \qquad \frac{1}{m}\sum_{j=l}^{l+m-1} x_j^2 \le P, \qquad \frac{1}{m}\sum_{j=l}^{l+m} x_j^2 = P$$

for all $1 \le l \le m + l \le n$. In this respect, the saturation requirement can be described as a *constant power* constraint. From this, it follows that

$$\mathfrak{P}_n \subset \mathfrak{P}_n^p \subset \mathfrak{P}_n^a;$$

this in turn implies

$$C_T \le C_T^p \le C_T^a \quad \text{and} \quad C \le C^p \le C^a.$$

The upper bound on the capacity C of the saturation recording channel is obtained by the average capacity bound C^a. This bound is obtained by maximizing the upper bound on the mutual information given by (7). The maximization involves the proper choice of input power spectral density, $S_x(e^{i2\pi f})$, subject to the average power constraint.

The lower bound is obtained by maximizing the lower bound on the mutual information given by (8). However, it is not hard to show that the differential entropy, $h(x) = -\infty$, for discrete random variables. This means that for distributions on the n-cube, such as those described by (9b), the lower bound (8) is trivially equal to 0 since $h(X) = -\infty$. Fortunately, the beautiful lemma 1 of [1] proves that if the channel impulse response is square integrable

$$\int_{-\infty}^{+\infty} h(t)^2\, dt < \infty$$

then the peak power capacity is equal to the capacity of the saturation recording channel

$$C = C^p.$$

Essentially, it is shown that for any square integrable impulse response $h(t)$ and any symbol period T, there is an integer L such that $C_{T/L} \approx C_T^p$. Thus, the capacity of the saturation recording channel can be lower bounded by finding a good lower bound to C_T^p. This is accomplished by maximizing the lower bound on the mutual information given by (8).

Without reproving lemma 1 of [1], the essence of the proof can be seen by the following loose argument. Consider any signal of the form of (1):

$$x(t) = \sum_{j=1}^{n} x_j B_T(t - jT)$$

where the symbols are *bounded* $-\sqrt{P} \le x_m \le +\sqrt{P}$. Assume the channel impulse response, $h(t)$, is square integrable. The signal component of the sampled output of the matched filter receiver, matched to $p_T(t)$, is the vector $M_T\boldsymbol{x}$. The $n \times n$ matrix, M_T is given by (4) in terms of samples of the autocorrelation function $R_{p_T}(\tau)$

$$M_T = \{m_{ij}\}_{0 \le i,j < n},$$

$$m_{ij} = \int_{-\infty}^{+\infty} p_T((i-j)T + t) p_T(t)\, dt.$$

Then given $M_T\boldsymbol{x}$, for any bounded $\boldsymbol{x}$, there is an integer L and a *saturation* signal

$$\hat{x}(t) = \sum_{j=1}^{Ln} \hat{x}_j B_{T/L}(t - jT/L)$$

(with symbol period T/L and symbols $\hat{x}_j \in \{-\sqrt{P}, +\sqrt{P}\}$) such that the n samples of the output of the filter matched to $p_T(t)$ is close to the vector $M_T\boldsymbol{x}$. Specifically, for any $\epsilon > 0$, there is an L (which depends only on $h(t)$ and ϵ and not $\boldsymbol{x}$ and n) such that the mean squared error

$$\|M_T\boldsymbol{x} - \hat{M}_{T/L}\hat{\boldsymbol{x}}\|^2 \le n\epsilon.$$

Here the $n \times nL$ matrix

$$\hat{M}_{T/L} = \{\hat{m}_{ij}\}_{1 \le i \le n, 1 \le j \le nL},$$

$$\hat{m}_{ij} = \int_{-\infty}^{+\infty} p_T((i-j)\,T + t)\, p_{T/L}(t)\, dt$$

is equal to samples of the *cross correlation* of $p_T(t)$ and $p_{T/L}(t)$.

This result says that for any bounded signal with period T, the sampled output of the matched filter can be accurately approximated by a saturation signal with a smaller symbol period T/L. In practice, this result suggests that it may be advantageous to use a *signaling period*, in the write process, which is smaller than the *sampling period* in the read process.

C. Bounds on Capacity

Bounds on the capacity of saturation recording are obtained by combining the upper and lower bounds on the mutual information with the results on the channel capacity.

The upper bound on the capacity follows from the fact that the capacity of the average power constrained channel bounds the capacity of the saturation channel, $C_T \le C_T^a$ and $C \le C^a$. The usefulness of this inequality follows from the fact that the capacity of the average power constrained channel is known and can be readily computed [8, Ch. 7 and 8]. The upper bound on the average mutual information (7) is achieved with equality by a Gaussian process with a specified power spectral density $S_x(e^{i2\pi f})$. Note that such processes generate random vectors $\boldsymbol{x}$ that satisfy a power constraint but not a saturation (or peak power) constraint.

The calculation of the capacity C_T^a becomes one of maximizing the upper bound (7)

$$I(X;\, Y) = \frac{1}{2}\int_0^1 \log\left(\frac{2S_x(e^{i2\pi f})\, S_{p_T}(e^{i2\pi f})}{N_o}\right) df$$

subject to the power constraint

$$\int_0^1 S_x(e^{i2\pi f})\, df = P.$$

The solution is known as *water filling*. Write the mutual information

$$I(X;\, Y) = \frac{1}{2}\int_0^1 \log\left(\frac{1}{S_{p_T}(e^{i2\pi f})} + \frac{2S_x(e^{i2\pi f})}{N_o}\right) df + \frac{1}{2}\int_0^1 \log\,(S_{p_T}(e^{i2\pi f}))\, df.$$

The second term is independent of the choice of the input spectrum $S_x(e^{i2\pi f})$. The first term is maximized by trying to make the expression that appears in the logarithm equal to a constant. This is not always possible, e.g., when the pulse power spectrum, $S_{p_T}(e^{i2\pi f})$, has a zero; this condition holds for the differentiated Lorentz channel with spectrum given by (5) since $S_{p_T}(e^{i2\pi f}) = 0$ for $f = 0$.

The water-filling solution is best described by the level parameter $0 < \lambda < \infty$. Define the set of frequencies

$$A_\lambda \equiv \{-\tfrac{1}{2} \le f \le \tfrac{1}{2} | \lambda S_{p_T}(e^{i2\pi f}) \ge 1\} \tag{10a}$$

where the level λ exceeds the inverse of the pulse spectrum $1/S_{p_T}(e^{i2\pi f})$. The optimal input spectrum, for the given level λ, is given by

$$S_x(e^{i2\pi f}) = \begin{cases} \dfrac{N_o}{2}\left(\lambda - \dfrac{1}{S_{p_T}(e^{i2\pi f})}\right), & f \in A_\lambda; \\ 0 & \text{otherwise.}\end{cases}$$

Then the capacity, in terms of the parameter λ,

$$C_T \le C_T^a = \frac{1}{2T}\int_{A_\lambda} \log\,(\lambda S_{p_T}(e^{i2\pi f}))\, df \tag{10b}$$

$$\rho \equiv \frac{P}{N_o} = \frac{1}{2}\int_{A_\lambda} \left(\frac{1}{S_{p_T}(e^{i2\pi f})}\right) df \tag{10c}$$

where ρ is the *channel signal-to-noise ratio*.

In the limit, as $T \to 0$, the channel capacity is bounded by

$$C \le C^a = \frac{1}{2}\int_{A_\lambda} \log\,(\lambda |H(f)|^2)\, df \tag{11a}$$

$$\rho = \frac{P}{N_o} = \frac{1}{2}\int_{A_\lambda} \left(\lambda - \frac{1}{|H(f)|^2}\right) df \tag{11b}$$

where $H(f)$ is the transfer function of the recording channel (i.e., the Fourier transform of the channel impulse response) and

$$A_\lambda \equiv \{-\infty < f < \infty \mid \lambda |H(f)|^2 \geq 1\}. \quad (11c)$$

The lower bound on the capacity of the saturation recording is obtained by finding a lower bound to C^p and by using the fact, proven in [1], that $C = C^p$. For a given symbol period, (8) provides a lower bound on the information per symbol in terms of the average differential entropy $h(X)$. In the peak power case, it is easy to show that:

$$h(X) \leq \log 2\sqrt{P}$$

with equality if and only if the input vector $\boldsymbol{x}$ is drawn i.i.d., uniform on the interval $[-\sqrt{P}, \sqrt{P}]$. Thus, substituting this best lower bound [(in terms of $h(X)$)]

$$C \geq C_T^p \geq \frac{1}{2T} \log \left(1 + \frac{4P}{\pi e N_o} 2^{\int_0^1 \log (S_{pT}(e^{i2\pi f}))df}\right) \frac{\text{bits}}{\text{second}}. \quad (12)$$

Note that while C^p is equal to the limit of C_T^p as $T \to 0$, the best lower bound (in terms of T) on the capacity will be provided for a positive symbol period T^*. (In fact, the lower bound approaches zero as the symbol period goes to zero.) Thus, the best lower bound on the capacity

$$C = C^p \geq \frac{1}{2T^*} \log \left(1 + \frac{4P}{\pi e N_o} 2^{\int_0^1 \log (S_{pT}(e^{i2\pi f}))df}\right) \frac{\text{bits}}{\text{second}} \quad (13)$$

where $T^* > 0$ is the symbol period that maximizes the lower bound (12).

Equations (10) and (11) provide upper bounds on the capacities C_T and C, respectively, while (13) provides a lower bound to the capacity C. Note that, while (12) lower bounds the peak power capacity C_T^p, it does not provide a lower bound to C_T. While the question of a good lower bound to C_T is not addressed here, progress on this problem can be found in [15].

III. Computing Bounds on the Capacity

A. Bounds on Capacity—Differentiated Lorentz Channel

The derived expressions (10), (11), (13), provide a method for obtaining bounds on the capacity of a particular recording channel with a given the channel impulse response $h(t)$. In the case of the differentiated Lorentz channel, we have from (5)

$$|H(f)|^2 = 4\pi^4 f^2 e^{-4\pi|f|}$$

and

$$S_{pT}(e^{i2\pi f}) = \frac{4\pi^2}{T \sinh\left(\frac{2\pi}{T}\right)} \sin^2(\pi f) \cdot \cosh\left(\frac{4\pi}{T}\left(|f| - \frac{1}{2}\right)\right), \quad -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

Using this model, the upper bounds to C_T and C and the lower bound to C are computed as a function of the channel signal-to-noise ratio ρ.

For example, to compute C^a for the differentiated Lorentz channel, we first note that $|H(f)|^2$ has a maximum at $f = 1/2\pi$ and that $|H(1/2\pi)|^2 = \pi^2/e^2$. Thus, the level parameter $e^2/\pi^2 < \lambda < \infty$. The function $|H(f)|^2$ is monotonically increasing on the interval $0 \leq f \leq 1/2\pi$ and monotonically decreasing for $1/2\pi \leq f \leq \infty$. Thus the water-filling frequencies are the union of two intervals

$$A_\lambda = [-f_1, -f_0] \cup [f_0, f_1]$$

where $0 < f_0 < 1/2\pi < f_1$ satisfy the equation

$$f_i = \sqrt{\frac{1}{4\pi^4 \lambda}} e^{2\pi f_i}, \qquad i = 0, 1.$$

Note that this equation can be iterated to find f_0 and f_1. Given the endpoints, it follows

$$C^a = 2 \log(e)[\pi(f_1^2 - f_0^2) - (f_1 - f_0)]$$

and

$$\pi = \frac{P}{N_o} = \lambda(f_1 - f_0) - \frac{1}{4\pi^4} \int_{1/f_1}^{1/f_0} e^{4\pi/x}\, dx.$$

This last integral can not be obtained in closed form and must therefore be computed numerically. Similarly, when C_T^a is to be computed, the water-filling frequencies are the union of two intervals described by $0 < f_0 < f_1 \leq \frac{1}{2}$ where $f_1 = \frac{1}{2}$ if $\lambda \geq T \sinh(2\pi/T)/4\pi^2$. Fig. 3 shows the water-filling spectrum for the case $\rho = 10$ (db) and $T^* = 1.14$ (the best symbol period for the lower bound). In this case, $\lambda = 22.46$.

The fourth figure (Fig. 4) shows the computed values of the upper and lower bounds to the channel capacity, C_1 as a function of the channel signal-to-noise ratio ρ.

The following figure (Fig. 5) depicts the upper and lower bounds to C_T^p as a function of the symbol period T for a few values of the channel signal-to-noise ratio ρ. Note that the upper bound on the saturation capacity C is obtained as the limit of the upper bound C_T^a as $T \to 0$. However, the lower bound to C is the maximum of the lower bound to C_T^p. This maximum is obtained for a positive symbol period T^* (i.e., T^* is the value of T where the lower bound curve peaks).

In numerically computing the lower bound (13), the in-

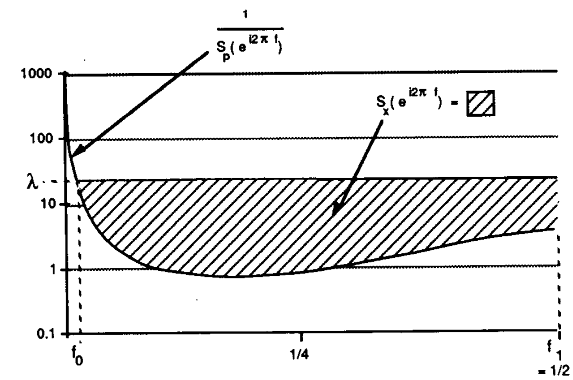
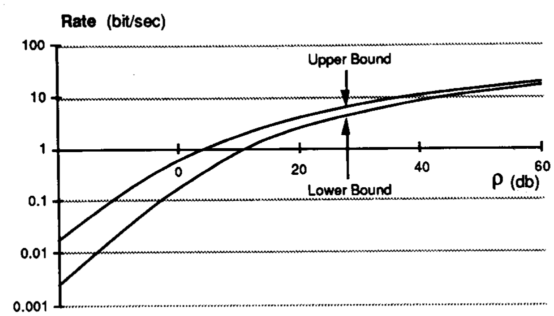


Fig. 3. Waterfilling differentiated Lorentz channel [$T^* = 1.14$, $\lambda = 22.46$, $\rho = 10$ (db)].



ρ	Upper Bound	T*	Lower Bound	ρ*/ρ
-20.	.0176	1.68	.0023	-9.69
-10.	.1349	1.66	.0223	-9.44
0.	.6589	1.51	.1850	-8.79
10.	1.922	1.13	.8922	-8.06
20.	3.968	0.82	2.399	-7.76
30.	6.731	0.62	4.661	-7.66
40.	10.17	0.50	7.621	-7.63
50.	14.27	0.42	11.25	-7.61
60.	18.95	0.36	15.54	-7.60
(db)	(bps)	(sec)	(bps)	(db)

Fig. 4. Capacity versus ρ, differentiated Lorentz channel.

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$$\int_0^1 \log\,(S_{p_T}(e^{i2\pi f}))\,df$$

needs to be estimated. This seems to pose a numerical problem since the pulse spectrum, $S_{p_T}(e^{i2\pi f})$, has a root at $f = 0$ ($4\pi \sin^2(2\pi f)$ is a factor) and $\log(0) = -\infty$. However, this factor can simply be ignored in the calculation since $\int_0^1 \log(4\pi \sin^2(2\pi f))\,df = 0$.

Also shown on Fig. 5 is the trivial upper bound

$$C_T \leq \frac{1}{T}.$$

This follows from the simple fact that the binary input signal, in (1), can convey at most one bit of information ($x_j \in \{+\sqrt{P}, -\sqrt{P}\}$).

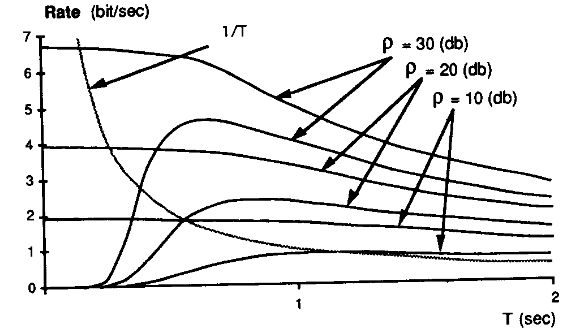

Fig. 5. Bounds versus T, differentiated Lorentz channel.

The best value of the symbol period, T^*, for the lower bound is displayed in Fig. 6 as a function of ρ.

Finally, the upper and lower bounds are compared in terms of a loss in signal-to-noise ratio in Fig. 7. The power loss is defined as follows. Given a value of signal-to-noise ratio ρ, first compute the lower bound on the capacity C. Let this lower bound rate be R. Then, the value of signal-to-noise ratio ρ^* required for the upper bound to meet the rate, R, of the lower bound is computed. The power loss is then defined as the ratio ρ^*/ρ. It is interesting to observe that the power loss approaches $2e/\pi^3 \approx -7.56$ (db) as $\rho \to \infty$. This is the same result that is obtained for high signal-to-noise ratio in the strictly lowpass examples computed in [1].

B. Bounds on Capacity—Applications

Bounds on capacity of saturation recording have been presented in Section III. The usefulness of the results follows from the fact that they can be computed from measurements on an actual recorder. In order to evaluate the bounds on capacity, estimates of the channel response and the operating signal-to-noise ratio are required.

In the saturation recording systems, the response of the system is often characterized in terms of the step response

$$g(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

of the channel. Note that, in terms of the step response, the filtered version of the input signal

$$\begin{aligned}\tilde{x}(t) &= h * x(t)\\ &= \sum_{j=1}^{n} x_j p_T(t - jT)\\ &= \sum_{j=1}^{n} x_j (g(t - jT) - g(t - (j+1)T))\\ &= \sum_{j=1}^{n} (x_j - x_{j-1}) g(t - jT).\end{aligned}$$

There are numerous methods for obtaining such estimates of $g(t)$. They range from the simple procedure of averaging the response to isolated transitions, to procedures

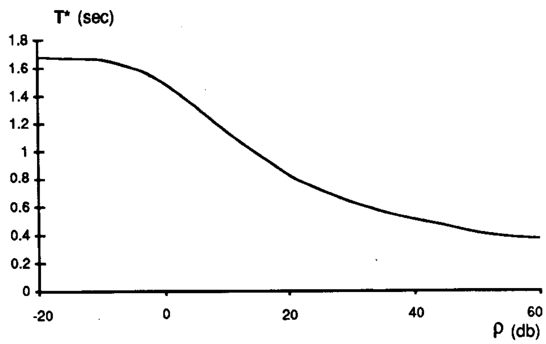

Fig. 6. T^* versus ρ, differentiated Lorentz channel.

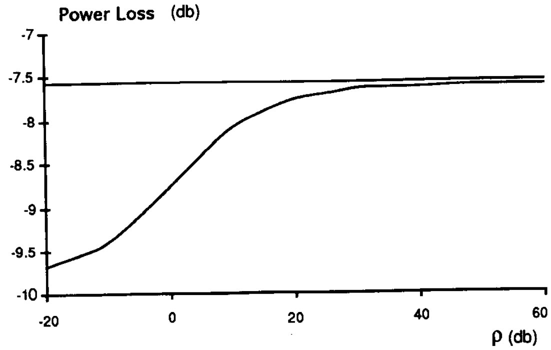

Fig. 7. Power loss versus ρ, differentiated Lorentz channel.

such as recursive least-squares approaches used in adaptive signal processing areas (see, for example, [13], [14]). These issues are beyond the scope of the present discussion. Given the response $g(t)$, the pulse response $p_T(t) = g(t) - g(t - T)$ is readily obtained and the bounds, such as Fig. 4, can be computed from (10), (11), and (13). This was done for the Lorentz example of Section III-D.

Once an estimate of the step response is determined, the channel signal-to-noise, ρ, can be estimated. Consider the situation of an isolated transition

$$x_k = \begin{cases} +\sqrt{P} & k \geq 0; \\ -\sqrt{P} & k < 0, \end{cases}$$

or

$$\tilde{x}(t) = 2\sqrt{P}g(t) + z(t).$$

If the output signal $y(t)$ is passed through a filter matched to the step response $g(t)$ and sampled at time 0, the following is observed:

$$(\tilde{x} + z) * g^-(0) = 2\sqrt{P}\,\|g\|^2 + w$$

where the step response energy

$$\|g\|^2 = \int_{-\infty}^{\infty} g(t)^2\,dt$$

and w is zero mean and has variance $N_o\|g\|^2/2$. By estimating the mean of the sampled output, $m = 2\sqrt{P}\|g\|^2$, and the variance, $\sigma^2 = N_o\|g\|^2/2$, the channel signal-to-noise ratio

$$\rho = \frac{P}{N_o} = \frac{m^2}{8\sigma^2\|g\|^2}$$

is obtained. This value of ρ can then be used in conjunction with the curves derived from the estimated step response to obtain the upper and lower bounds on the capacity. Note that scale factors involving the estimated step response cancel. For example, if the estimated response, $\tilde{g}(t) = 2g(t)$, then the ρ-axis of the capacity curve (e.g., Fig. 3) is shifted by 6 (db). However, this effect is canceled when the signal-to-noise ratio is estimated since $(m^2/8\sigma^2\|\tilde{g}\|^2)$ will be 6 (db) lower. This follows from the fact that the ratio $m^2/\sigma^2 = 8P\|g\|^2/N_o$ does not depend on the scaling of $\tilde{g}(t)$.

Appendix

The expression for the pulse spectrum, $S_p(e^{i2\pi f})$, for the general Lorentz model, $p_T(t) = g_L(t) - G_L(t - T)$, follow from the following identities. Let $\gamma \equiv e^{2\pi\alpha}$, $\beta \equiv e^{i2\pi\tau}$, then

$$\gamma^x + \gamma^{-x} = 2\cosh(2\pi\alpha x), \qquad \gamma^x - \gamma^{-x} = 2\sinh(2\pi\alpha x)$$

$$\beta^x + \beta^{-x} = 2\cos(2\pi\tau x), \qquad \beta^x - \beta^{-x} = 2i\sin(2\pi\tau x)$$

$$A(\gamma) \equiv \sum_{l=-\infty}^{\infty} \gamma^{|f-l|} = \frac{\gamma^{f-1/2} + \gamma^{-f+1/2}}{\gamma^{1/2} - \gamma^{-1/2}} = \frac{\cosh(2\pi\alpha(f - 1/2))}{\sinh(\pi\alpha)}$$

$$B(\beta, \gamma) \equiv \sum_{l=-\infty}^{\infty} \mathrm{sgn}(f - l)\beta^{f-l}\gamma^{-|f-l|} = \frac{\beta^f(\gamma^{f-1} + \gamma^{1-f}) - \beta^{f-1}(\gamma^f + \gamma^{-f})}{(\gamma^{1/2} - \gamma^{-1/2}) - (\beta - \beta^{-1})}$$

$$B(\beta, \gamma) - B(\beta^{-1}, \gamma) = \frac{(\beta^f - \beta^{-f})(\gamma^{f-1} + \gamma^{1-f}) - (\beta^{f-1} - \beta^{1-f})(\gamma^f + \gamma^{-f})}{(\gamma^{1/2} - \gamma^{-1/2}) - (\beta - \beta^{-1})}$$

$$= 2i\,\frac{\sin(2\pi\tau f)\cosh(2\pi\alpha(1 - f)) + \sin(2\pi\tau(1 - f))\cosh(2\pi\alpha f)}{\cosh(2\pi\alpha) - \cosh(2\pi\tau)}.$$

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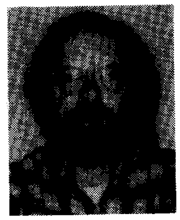

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