Binary Convolutional Codes with Application to Magnetic Recording

A. R. CALDERBANK, CHRIS HEEGARD AND TING-ANN LEE

Abstract—Calderbank, Heegard, and Ozarow [1] have suggested a method of designing codes for channels with intersymbol interference, such as the magnetic recording channel. These codes are designed to exploit intersymbol interference. The standard method is to minimize intersymbol interference by constraining the input to the channel using run-length limited sequences. Calderbank, Heegard, and Ozarow considered an idealized model of an intersymbol interference channel that leads to the problem of designing codes for a partial response channel with transfer function $(1 - D^N)/2$, where the channel inputs are constrained to be ± 1. This problem is considered here. Channel inputs are generated using a nontrivial coset of a binary convolutional code. The coset is chosen to limit the zero-run length of the output of the channel and so maintain clock synchronization. The minimum squared Euclidean distance between outputs corresponding to distinct inputs is bounded below by the free distance of a second convolutional code which we call the magnitude code. An interesting feature of the analysis is that magnitude codes that are catastrophic may perform better than those that are noncatastrophic.

I. Introduction

A CODING TECHNIQUE suitable for high-density magnetic recording has been described by Calderbank, Heegard and Ozarow in [1]. Simple recording codes are presented that allow an increase in recording density and that decrease the probability of error when the information stored on the disk or tape is retrieved. The codes are designed to *exploit intersymbol interference* in the magnetic recording channel. Decoding is accomplished by maximum likelihood sequence estimation which is implemented using the Viterbi algorithm.

Modulation codes designed under the assumption of a peak detector use run-length limited sequences to guarantee a minimum (and maximum) separation between the peaks of the read signal (the maximum separation is provided for clock synchronization). These modulation codes seek to *minimize intersymbol interference* by constraining the input to the channel. The recent papers by Adler, Coppersmith, and Hassner [2] and by Schouhamer Immink [3], [4], [5] offer two rather different approaches to the design of this type of modulation code.

The model of the intersymbol interference channel studied by Calderbank, Heegard, and Ozarow has three significant attributes.

1) The writing current in the recording head is sufficient to ensure positive or negative saturation of the magnetic medium. Thus it is only possible to write the symbols ± 1.
2) The recording process differentiates and low-pass filters the input waveform. Differentiation occurs because we assume that the read head detects changes in the pattern of magnetization.
3) The filtered waveform is corrupted by additive noise.

The approach applies (in principle) to any finite duration step response but the authors focus on two idealized step responses. One of these is a true moving average or "square" response $p(t) = B_1(t)$, where

$$B_T(t) = \begin{cases} 1, & \text{if } -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}.$$

The authors argue that at symbol rate N (the write signal $x(t) = \sum_{j=0}^{\infty} \alpha_j B_T(t - jT)$ where $T = 1/N$ and $\alpha_j = \pm 1$) the probability of decoder error behaves as $Q(\sqrt{d^2A^2/N})$, where

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

is the tail of the unit Gaussian distribution, A is the system gain, and

$$d^2 = \min_{x \ne x^*} \left\| \frac{1}{2}(1 - D^N)(x - x^*) \right\|^2,$$

where the minimum is taken over all pairs of distinct codewords (channel inputs) corresponding to messages that agree in all but a finite number of places. This motivates the design of simple trellis codes for channels with transfer functions $(1 - D^N)/2$ where channel inputs are constrained to be ± 1.

For $N > 3$ this model is not applicable to the magnetic recording channel. For $N = 1, 2$ it coincides with models that have been studied previously in the magnetic recording literature (see Kobayashi and Tang [6], Nakagawa, Yokoyama, and Katayama [7], Wood, Ahlgrim,

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Hallamasek, and Stenerson [8]). But in any case the problem of code design is of interest in its own right.

The encoders that we consider here transform a binary message into a codeword using a binary convolutional code. We design codes that increase the minimum squared Euclidean distance d^2 between outputs corresponding to distinct inputs. We generate the channel inputs using a non-trivial coset of the convolutional code. This eliminates long sequences of zeros as possible channel outputs and so maintains clock synchronization. The most interesting aspect of the analysis is perhaps that encoders leading to "catastrophic" binary convolutional codes perform better than encoders leading to noncatastrophic codes on the $(1 - D^N)/2$ channel. We advise the reader to begin with the examples presented in Section II since they illuminate the theory presented in the rest of the paper. Section III formalizes the problem of code design within an appropriate algebraic framework. Section IV discusses methods of code construction. The problem of clock synchronization is solved in Section V. Theorems covering the overall performance of rate $k/(k+1)$ codes on the $(1 - D)/2$ and $(1 - D^2)/2$ channel are presented in Section VI. Section VII is a list of recording codes and their parameters.

II. Some Simple Recording Codes

In this section we present simple trellis codes for partial response channels with transfer functions $(1 - D^N)/2$ where the channel inputs are constrained to be ± 1. Calderbank, Heegard, and Ozarow introduced this type of trellis code in [1]. The encoder transforms a binary message into a codeword (sequence of channel inputs) using a binary convolutional code. The minimum squared Euclidean distance between outputs corresponding to distinct inputs determines the probability of decoder error. This distance is bounded below by the free distance of a second binary convolutional code. The transfer function of the partial response channel determines the relationship between these two convolutional codes.

The idea of lower bounding the Euclidean distance of a trellis code by the Hamming distance of a convolutional code is described by Ungerboeck [9] and by Calderbank and Mazo [10]. It is also used by Wolf and Ungerboeck in [11].

A second purpose served by the trellis code is to provide clock synchronization. If long sequences of zeros are possible channel outputs then the decoder will have difficulty maintaining synchronization. Lee and Heegard [12] demonstrate that it is possible to limit the zero-run length of the output by adding a fixed periodic binary sequence to each codeword.

Example 1: This is the simplest possible example; a rate $1/2$ code for the $(1 - D^2)/2$ channel requiring a two-state decoder. The repetition code $[1,1]$ is used to generate channel inputs. The encoder receives a message $\boldsymbol{m} = (m_0, m_1, m_2, \cdots)$, $m_i = \pm 1$, and transmits a codeword $\boldsymbol{x} = (x_0, x_1, x_2, \cdots)$, $x_i = \pm 1$. At time k the encoder generates code bits x_{2k}, x_{2k+1} by the rule

$$x_{2k} = x_{2k+1} = m_k.$$

(The correspondence between the 0,1 world and the ± 1 world is quite straightforward; modulo 2 addition in the 0,1 world corresponds to multiplication in the ± 1 world.) The decoder reconstructs the message $\boldsymbol{m}$ by comparing the corrupted channel output with estimates of the sequence $\boldsymbol{s} = (1 - D^2)\boldsymbol{x}/2$, where

$$s_{2k} = s_{2k+1} = \frac{1}{2}(m_k - m_{k-1}).$$

The decoder requires two states. Let $\boldsymbol{w} \neq \boldsymbol{m}$ be a message, let $\boldsymbol{x}^*$ be the codeword corresponding to $\boldsymbol{w}$ and let $\boldsymbol{s}^* = (1 - D^2)\boldsymbol{x}^*/2$. If exactly one of the pairs (w_k, m_k), (w_{k-1}, m_{k-1}) are not identical then

$$|s_{2k} - s_{2k}^*| = |s_{2k+1} - s_{2k+1}^*| = 1.$$

If

$$d^2 = \min_{\boldsymbol{x} \neq \boldsymbol{x}^*} \left\{ \left\| \frac{1}{2}(1 - D^2)(\boldsymbol{x} - \boldsymbol{x}^*) \right\|^2 \right\},$$

then d^2 is bounded below by the free distance of the rate $1/2$ binary convolutional code with generator matrix $[1 + D, 1 + D]$. Fig. 1 is the state diagram of this code. The labels on the two states correspond to the possible values of the prior message bit. An edge represents a transition between states and is labelled with the Hamming weight of the corresponding output. The relationship between the convolutional code used to encode messages and the convolutional code used to determine the performance of the decoder is given by

$$[1 \quad 1]\begin{bmatrix} (1+D) & 0 \\ 0 & (1+D) \end{bmatrix} = [1 + D \quad 1 + D].$$

The matrix $(1 + D)I$ is called the *channel matrix*. It represents the operation of adding (modulo 2) a sequence of 2-bit bytes to a 2-bit shift ($N = 2$) of itself.

We introduce the symbol $\approx$ to mean "agrees in all but a finite number of places." Consider two codewords $\boldsymbol{x}, \boldsymbol{x}'$ corresponding to messages $\boldsymbol{m}, \boldsymbol{m}'$ respectively. If $\boldsymbol{x} \not\approx \boldsymbol{x}'$ and $(1 - D^2)\boldsymbol{x}/2 \approx (1 - D^2)\boldsymbol{x}'/2$ then the decoder will be unable to distinguish $\boldsymbol{x}$ and $\boldsymbol{x}'$. Write $m_j = (-1)^{y_j} m_j'$ where $y_j = 0$ or 1. Then

$$\frac{1}{2}(1 - D^2)\boldsymbol{x} \approx \frac{1}{2}(1 - D^2)\boldsymbol{x}' \pmod 2$$
$$\Leftrightarrow \boldsymbol{y}[1 + D, 1 + D] \approx [\mathbf{0}, \mathbf{0}],$$

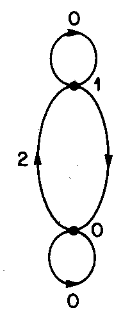

Fig. 1. State diagram of binary convolutional code with generator matrix $[1 + D, 1 + D]$.

and so $y \approx 1/(1+D) = (1,1,1,\cdots)$, $m' \approx -m$, and $x' = -x$. If $(1-D^2)x/2 \approx (1-D^2)(-x)/2$ then $(1-D^2)x \approx \mathbf{0}$ and $m_k = m_{k-1}$ for all but finitely many k. The only pair of messages that the decoder cannot distinguish are $\pm(1,1,\cdots)$.

A second problem with the constant message is that the decoder clock requires transitions in the output sequence s to maintain synchronization, and a constant message sequence produces the zero output sequence. Again this problem is associated only with constant message sequences.

The remedy is very simple; define

$$x_{2k} = (-1)^k m_k, \; x_{2k+1} = m_k.$$

Then

$$s_{2k} = \frac{(-1)^k}{2}(m_k + m_{k-1}), \; s_{2k+1} = \frac{1}{2}(m_k - m_{k-1}).$$

The Euclidean distance d^2 is still bounded below by the free distance of the code $[1+D, 1+D]$, which is 4. However there is no message m for which $(1-D^2)x/2 \approx (1-D^2)(-x)/2$; indeed the maximum zero-run length of an output sequence is just 2.

Example 2: In Example 1 we designed a rate 1/2 code for the $(1-D^2)/2$ channel. It is of course possible to regard the $(1-D^2)/2$ channel as two interleaved $(1-D)/2$ channels and to design a rate 1/2 code for the $(1-D)/2$ channel. This example is designed to illustrate the limitations of this approach.

The code $[1+D+D^2, 1]$ is used to generate channel inputs and we refer to this code as the sign code (following the terminology introduced by Lee and Heegard [9]). The encoder receives message bits $m_k = \pm 1$, and at time k, generates code bits x_{2k}, x_{2k+1} by the rule

$$x_{2k} = m_k m_{k-1} m_{k-2}, \; x_{2k+1} = m_k.$$

The output sequence $s = (1-D)x/2$ is given by

$$s_{2k} = \frac{1}{2}(m_k m_{k-1} m_{k-2} - m_{k-1}),$$

$$s_{2k+1} = \frac{1}{2}(m_k - m_k m_{k-1} m_{k-2}).$$

The decoder requires 4 states. Let $w \neq m$ be a message, let x^* be the codeword corresponding to w, and let $s^* = (1-D^2)x^*/2$. If an odd number of the pairs (m_k, w_k), (m_{k-2}, w_{k-2}) are different, then $|s_{2k} - s^*_{2k}| = 1$. If an odd number of the pairs (m_{k-1}, w_{k-1}), (m_{k-2}, w_{k-2}) are different, then $|s_{2k+1} - s^*_{2k+1}| = 1$. Thus

$$d^2 = \min_{x \neq x^*} \left\{ \left\| \frac{1}{2}(1-D)(x - x^*) \right\|^2 \right\}$$

is bounded below by the free distance of a binary convolutional code with generator matrix $[1+D^2, D+D^2]$. Fig. 2 is the state diagram of this code. Again we adopt the terminology introduced by Lee and Heegard [12], and we refer to the code $[1+D^2, D+D^2]$ as the magnitude code. The magnitude code is obtained by multiplying the sign code by the channel matrix which represents the operation of adding (modulo 2) a sequence of 2-bit bytes to a 1-bit shift of itself:

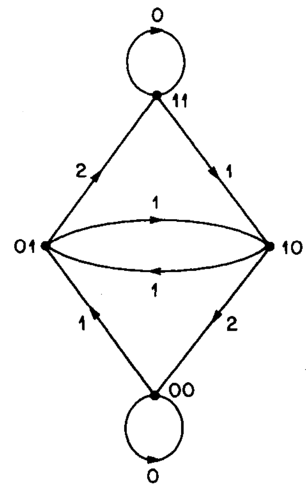

Fig. 2. State diagram of binary convolutional code with generator matrix $[1+D^2, D+D^2]$.

$$[1+D+D^2, 1]\begin{bmatrix} 1 & 1 \\ D & 1 \end{bmatrix} = [1+D^2, D+D^2].$$

The magnitude code results from using the sign code to generate inputs to a modulo 2 (0,1) channel with transfer function $1+D$.

Again we need to identify all pairs of codewords x, x' such that $x \not\approx x'$ and $(1-D)x/2 \approx (1-D)x'/2$. Suppose x, x' correspond to messages m, m' and write $m'_j = (-1)^{y_j} m_j$ where $y_j = 0$ or 1. Then

$$\frac{1}{2}(1-D)x \approx \frac{1}{2}(1-D)x' \pmod 2$$

$$\Leftrightarrow y[1+D^2, D+D^2] \approx [\mathbf{0}, \mathbf{0}].$$

Hence $y \approx 1/(1+D)$, $m' \approx -m$, and $x' \approx -x$ (since each product occurring in the formulas for x_{2k+j}, $j = 0, 1$, involves an odd number of terms m_l). Now $(1-D)x/2 \approx \mathbf{0}$ forces $m \approx \pm(1,1,1,\cdots)$. The only pair of messages that the decoder fails to distinguish is $\pm(1,1,1,\cdots)$. (This is not happenstance. We shall consider codewords x, x' generated by a rate k/n sign code for the $1-D$ channel such that $x \not\approx x'$ and $(1-D)x/2 \approx (1-D)x'/2$. We prove that $x' = -x$ and that after applying some nonsingular transformation to the inputs we may suppose $m'^j \approx -m^j$ for some $j \in \{0,1,\cdots,k-1\}$ and $m'^i \approx m^i$ for $i \neq j$.)

Again the remedy is very simple; define

$$x_{2k} = -m_k m_{k-1} m_{k-2}, \; x_{2k+1} = m_k.$$

Then

$$s_{2k} = -\frac{1}{2}(m_k m_{k-1} m_{k-2} + m_{k-1})$$

$$s_{2k+1} = \frac{1}{2}(m_k + m_k m_{k-1} m_{k-2}).$$

The free distance of the new code is still 4. However it is easily checked that there is no message m for which $(1-D)x/2 \approx (1-D)(-x)/2$; indeed the maximum zero-run length in an output sequence is 4.

Examples 1 and 2 feature rate 1/2 codes with squared Euclidean distance $d^2 = 4$ that can both be used on the $(1 - D^2)/2$ channel. Example 1 requires a 2-state decoder operating on a time-varying trellis with period 2 (the expression $s_{2k} = (-1)^k(m_k + m_{k-1})$ involves $(-1)^k$). Example 2 requires two 4-state decoders working in parallel on the same stationary trellis. The maximum zero-run length in Example 2 is 4 whereas in Example 1 it is 2.

Example 3: A rate 2/3 code for the $(1 - D)/2$ channel. This example is designed to introduce the reader to rate k/n sign codes, where $k \neq 1$, and to demonstrate the method of calculating the maximum zero-run length.

The generator matrix for the sign code is

$$G = \begin{bmatrix} 1 & D & 1 \\ 0 & 1 & D \end{bmatrix}.$$

The encoder receives messages $\boldsymbol{m}^i = (m_0^i, m_1^i, \cdots)$, $m_j^i = \pm 1$, $i = 0, 1$, and transmits a codeword $\boldsymbol{x} = (x_0, x_1, \cdots)$. At time k the encoder receives message bits m_k^0, m_k^1 and generates code bits $x_{3k}, x_{3k+1}, x_{3k+2}$ according to the rule

$$x_{3k} = m_k^0,\ x_{3k+1} = m_{k-1}^0 m_k^1,\ x_{3k+2} = m_k^0 m_{k-1}^1.$$

The output sequence $\boldsymbol{s} = (1 - D)\boldsymbol{x}/2$ is given by

$$s_{3k} = \frac{1}{2}\left(m_k^0 - m_{k-1}^0 m_{k-2}^1\right),$$

$$s_{3k+1} = \frac{1}{2}\left(m_{k-1}^0 m_k^1 - m_k^0\right),$$

$$s_{3k+2} = \frac{1}{2}\left(m_k^0 m_{k-1}^1 - m_{k-1}^0 m_k^1\right).$$

The decoder requires 8 states. Let $\boldsymbol{w}^i$, $i = 0, 1$, be messages, let $\boldsymbol{x}^*$ be the corresponding codeword, and let $\boldsymbol{s}^* = (1 - D)\boldsymbol{x}^*/2$. If an odd number of the pairs (m_k^0, w_k^0), (m_{k-1}^0, w_{k-1}^0), (m_{k-2}^1, w_{k-2}^1) are different, then $|s_{3k} - s_{3k}^*| = 1$. If an odd number of the pairs (m_{k-1}^0, w_{k-1}^0), (m_k^1, w_k^1), (m_k^0, w_k^0) are different, then $|s_{3k+1} - s_{3k+1}^*| = 1$. If an odd number of the pairs (m_k^0, w_k^0), (m_{k-1}^1, w_{k-1}^1), (m_{k-1}^0, w_{k-1}^0), (m_k^1, w_k^1) are different, then $|s_{3k+2} - s_{3k+2}^*| = 1$. Thus

$$d^2 = \min_{\boldsymbol{x} \neq \boldsymbol{x}^*} \left\{ \left\| \frac{1}{2}(1 - D)(\boldsymbol{x} - \boldsymbol{x}^*) \right\|^2 \right\}$$

is bounded below by the free distance of the binary convolutional code with generator matrix

$$F = \begin{bmatrix} 1 + D & 1 + D & 1 + D \\ D^2 & 1 & 1 + D \end{bmatrix}.$$

This is the magnitude code and it is obtained by multiplying the sign code by the channel matrix which represents the operation of adding a sequence of 3-bit bytes to a 1-bit shift of itself:

$$\begin{bmatrix} 1 & D & 1 \\ 0 & 1 & D \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ D & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + D & 1 + D & 1 + D \\ D^2 & 1 & 1 + D \end{bmatrix}.$$

The magnitude code results from using the sign code to generate inputs to a modulo 2 channel with transfer function $1 + D$. The magnitude code is catastrophic; the greatest common divisor of the determinants of the 2×2 minors of F is $1 + D$.

We identify all pairs of codewords $\boldsymbol{x}, \boldsymbol{x}'$ such that $\boldsymbol{x} \not\approx \boldsymbol{x}'$ and $(1 - D)\boldsymbol{x}/2 \approx (1 - D)\boldsymbol{x}'/2$. The only nonzero solution $[\boldsymbol{y}^0, \boldsymbol{y}^1]$ to

$$[\boldsymbol{y}^0, \boldsymbol{y}^1] \begin{bmatrix} 1 & D & 1 \\ 0 & 1 & D \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ D & 0 & 1 \end{bmatrix}$$

$$= [\boldsymbol{y}^0, \boldsymbol{y}^1] \begin{bmatrix} 1 + D & 1 + D & 1 + D \\ D^2 & 1 & 1 + D \end{bmatrix} \approx [\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}]$$

is $\boldsymbol{y}^0 \approx (1, 1, 1, \cdots)$, $\boldsymbol{y}^1 \approx \boldsymbol{0}$. Therefore the codewords $\boldsymbol{x}, \boldsymbol{x}'$ correspond to messages $[\boldsymbol{m}^0, \boldsymbol{m}^1]$ and $[-\boldsymbol{m}^0, \boldsymbol{m}^1]$ respectively. Since each product occurring in the formulas for x_{3k+j}, $j = 0, 1, 2$, involves an odd number of terms m_l^0, we have $\boldsymbol{x}' = -\boldsymbol{x}$. We still have to identify all messages $\boldsymbol{m}^0, \boldsymbol{m}^1$ for which $(1 - D)\boldsymbol{x}/2 \approx \boldsymbol{0}$. We do this by writing $m_k^i = (-1)^{y_k^i}$, where $y_k^i = 0$ or 1. Setting $s_{3k} = s_{3k+1} = s_{3k+2} = 0$ for all k, we observe that again we are looking for a solution to the equation

$$[\boldsymbol{y}^0, \boldsymbol{y}^1] F \approx [\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}],$$

The possible solutions are

$\boldsymbol{y}^0 = \boldsymbol{y}^1 = \boldsymbol{0}$; $\boldsymbol{m}^0 = \boldsymbol{m}^1 = (1, 1, 1, \cdots)$,

$\boldsymbol{y}^0 = \boldsymbol{1}$, $\boldsymbol{y}^1 = \boldsymbol{0}$; $\boldsymbol{m}^0 = (-1, -1, \cdots)$, $\boldsymbol{m}^1 = (1, 1, \cdots)$.

There are only 2 message pairs that the decoder fails to separate.

If we modify the encoding rules by taking

$$x_{3k} = m_k^0,\ x_{3k+1} = m_{k-1}^0 m_k^1,\ x_{3k+2} = -m_k^0 m_{k-1}^1,$$

then

$$s_{3k} = \frac{1}{2}\left(m_k^0 + m_{k-1}^0 m_{k-2}^1\right),$$

$$s_{3k+1} = \frac{1}{2}\left(m_{k-1}^0 m_k^1 - m_k^0\right),$$

$$s_{3k+2} = -\frac{1}{2}\left(m_k^0 m_{k-1}^1 + m_{k-1}^0 m_k^1\right).$$

Again write $m_k^i = (-1)^{y_k^i}$, where $y_k^i = 0$ or 1. Observe that the codeword $\boldsymbol{x}$ corresponding to messages $\boldsymbol{m}^0, \boldsymbol{m}^1$ satisfies $(1 - D)\boldsymbol{x}/2 \approx (0, 0, \cdots)$ if and only if the sequences $\boldsymbol{y}^0 = y^0(D)$, $\boldsymbol{y}^1 = y^1(D)$ satisfy

$$[y^0(D), y^1(D)] F \approx \left[\frac{1}{1 + D}, 0, \frac{1}{1 + D}\right].$$

We have to show that there is no codeword in the magnitude code that agrees with $(1, 0, 1)/(1 + D)$ in all but a finite number of places. A syndrome former for F with minimal memory is the 1×3 matrix

$$\frac{1}{1 + D}[F_1, F_2, F_3] = [D, 1 + D + D^2, 1 + D^2],$$

where F_i is the 2×2 minor obtained from F by suppressing column i. Since the inner product

$$\frac{1}{1 + D}(1, 0, 1)(D, 1 + D + D^2, 1 + D^2)^T = \frac{1 + D + D^2}{1 + D}$$

is not a polynomial, there is indeed no codeword in the magnitude code that agrees with $(1,0,1)/(1+D)$ in all but a finite number of places. To find the maximum zero-run length we use the coefficients of the polynomial entries of the syndrome former to create an array as shown below

$$\begin{array}{cccc|ccc|cccl} \boldsymbol{v} = (0 & 1 & 1 & & 1 & 1 & 0 & 0 & 1 & 1) & \\ & & D^2 & & & & & & & 1 & = 1 + D^2 \\ & D^2 & & & & D & & & 1 & & = 1 + D + D^2 \\ & & & & D & & & & & & = D \end{array}$$

A vector $\boldsymbol{c} = (c_1, c_2, \cdots, c_9)$ can be extended to a codeword in the magnitude code if and only if $\boldsymbol{c}\boldsymbol{v}^T = 0$. Now

$$\boldsymbol{w} = *11|\underbrace{101|101|100}_{9}|*$$

is a codeword in the magnitude code that agrees with $(1,0,1)/(1+D)$ in nine positions. This is the maximum possible.

Example 4: A rate 1/2 code for the $(1-D^2)/2$ channel. This example demonstrates that it is not always possible to eliminate all bad message pairs by modifying the encoding rules.

The generator matrix for the sign code is

$$G = [1 + D + D^2, 1 + D^2].$$

At time k, the encoder receives the message bit $m_k = \pm 1$ and generates code bits x_{2k}, x_{2k+1} according to the rule

$$x_{2k} = m_k m_{k-1} m_{k-2}, \; x_{2k+1} = m_k m_{k-2}.$$

The output sequence $\boldsymbol{s} = (1 - D^2)\boldsymbol{x}/2$ is given by

$$s_{2k} = \frac{1}{2}(m_k m_{k-1} m_{k-2} - m_{k-1} m_{k-2} m_{k-3}),$$

$$s_{2k+1} = \frac{1}{2}(m_k m_{k-2} - m_{k-1} m_{k-3}).$$

The generator matrix of the magnitude code is

$$F = [1 + D^3, 1 + D + D^2 + D^3]$$

and the free distance of this code is 6. The state diagram of this code is shown below in Fig. 3.

We identify all pairs of codewords $\boldsymbol{x}, \boldsymbol{x}'$ such that $\boldsymbol{x} \not\approx \boldsymbol{x}'$ and $(1-D^2)\boldsymbol{x}/2 \approx (1-D^2)\boldsymbol{x}'/2$. The method used in Examples 1, 2, and 3 shows that $\boldsymbol{x}, \boldsymbol{x}'$ correspond to messages $\boldsymbol{m}, -\boldsymbol{m}$, and that $x'_{2j} = -x_{2j}, x'_{2j+1} = x_{2j+1}$ for all but finitely many j. Next we find all messages $\boldsymbol{m}$ for which $[(1-D^2)\boldsymbol{x}/2]_{2j} = 0$ for all but finitely many j.

The only constraint on the message $\boldsymbol{m}$ is that $s_{2k} = 0$ for all but finitely many k. Hence $m_k = m_{k-3}$ for all but finitely many k and there are four bad pairs of messages $\pm\boldsymbol{m}$.

It is not possible to eliminate the bad message pairs by changing the encoding rules as in Examples 1, 2, and 3. Suppose we change the encoding rule to

$$x_{2k} = e_{2k} m_k m_{k-1} m_{k-2}, \; x_{2k+1} = e_{2k+1} m_k m_{k-2},$$

where $e_i = \pm 1$. Then messages $\boldsymbol{m}$ for which $(1-D)\boldsymbol{x}/2 \approx (1-D)(-\boldsymbol{x})/2$ satisfy

$$m_k = e_{2k} e_{2k-2} m_{k-3}$$

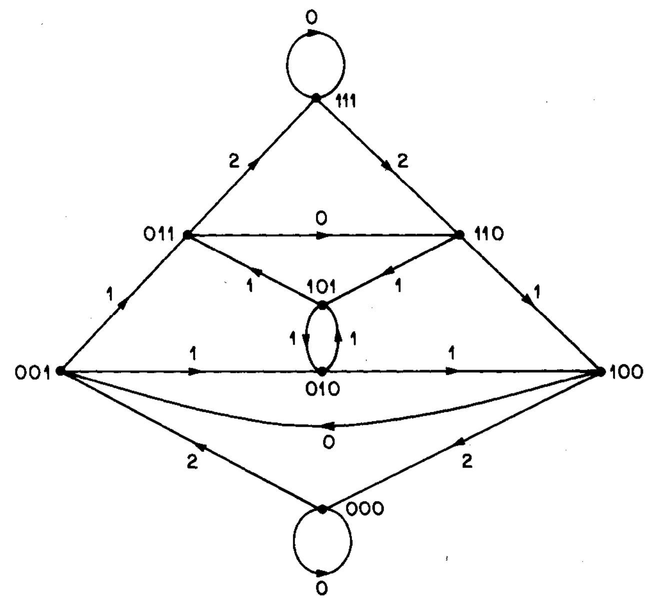

Fig. 3. State diagram of the rate 1/2 binary convolutional code with generator matrix $[1 + D^3, 1 + D + D^2 + D^3]$.

for all but finitely many k. Changing the encoding rule in this way changes the description of the bad message pairs but it does not change the number of bad pairs. However, the bad message pairs can be avoided by constraining the input to the encoder. The loss in data rate can be made arbitrarily small.

It remains to remove long sequences of zeros as possible channel outputs. We modify the encoding rules as follows:

$$x_{2k} = m_k m_{k-1} m_{k-2}, \; x_{2k+1} = (-1)^k m_k m_{k-2}.$$

Thus

$$s_{2k} = \frac{1}{2}(m_k m_{k-1} m_{k-2} - m_{k-1} m_{k-2} m_{k-3}),$$

$$s_{2k+1} = \frac{(-1)^k}{2}(m_k m_{k-2} + m_{k-1} m_{k-3}).$$

As in Example 3, we write $m_k = (-1)^{y_k}$, where $y_k = 0$ or 1. The codeword $\boldsymbol{x}$ determined by $\boldsymbol{m}$ satisfies $(1-D)\boldsymbol{x}/2 \approx (0,0,\cdots)$ if and only if the sequence $\boldsymbol{y} = y(D)$ satisfies

$$[y(D)][1 + D^3, 1 + D + D^2 + D^3] \approx \left[0, \frac{1}{1+D}\right].$$

We must show that there is no codeword in the magnitude code that agrees with $[0,1]/(1+D)$ in all but finitely many places. A syndrome former for F with minimal memory is

$$\frac{1}{1+D}[1 + D + D^2 + D^3, 1 + D^3] = [1 + D^2, 1 + D + D^2].$$

Since the inner product

$$\frac{1}{1+D}[0,1][1 + D^2, 1 + D + D^2]^T = \frac{1 + D + D^2}{1+D}$$

is not a polynomial, this modification of the encoding rule serves to remove arbitrarily long sequences of zeros as channel outputs. To find the maximum zero-run length we write down a binary vector v from the coefficients of the polynomial entries of the syndrome former:

$$\begin{array}{cc|cc|cc|l} v=(1 & 1 & 0 & 1 & 1 & 1) & \\ D^2 & & & & 1 & & =1+D^2 \\ & D^2 & & D & & 1 & =1+D+D^2 \end{array}$$

A vector $c = (c_1, c_2, \cdots, c_6)$ can be extended (in both directions) to a codeword in the magnitude code if and only if $cv^T = 0$. There is a codeword

$$w = *|\underbrace{11|01|01|00}_{6}|*$$

in the magnitude code that agrees with $(0,1)/(1+D)$ in six positions. This is the maximum possible.

Example 5: A rate 1/2 code for the $(1-D)/2$ channel. The performance of the recording codes described in Examples 1, 2, 3, and 4 is determined by the minimum squared Euclidean distance between outputs corresponding to distinct inputs. The free distance of the magnitude code is a lower bound on this Euclidean distance. The next example is a very simple and widely used recording code (biphase) that demonstrates that the lower bound is not always met with equality.

The generator matrix for the sign code is $G = [1,1]$. At time k the encoder receives the message bit $m_k = \pm 1$ and generates code bits x_{2k}, x_{2k+1} according to the rule

$$x_{2k} = -m_k, x_{2k+1} = m_k.$$

The output sequence $s = (1-D)x/2$ is given by

$$s_{2k} = -\frac{1}{2}(m_k + m_{k-1}), s_{2k+1} = m_k.$$

The generator matrix of the magnitude code is $F = [1 + D, 0]$ and the free distance of this code is 2. The minimum Euclidean distance $d^2 = 6$; if $m \approx m'$ are messages with $m_k \neq m'_k$ then $|s_{2k+1} - s'_{2k+1}|^2 = 4$, $|s_{2i} - s'_{2i}|^2 = 1$ for some $i \leq k$, and $|s_{2j} - s'_{2j}|^2 = 1$ for some $j > k$. The maximum zero-run length is 1.

III. An Algebraic Model

In Section 2 we described five recording codes for partial response channels with transfer functions $(1 - D^N)/2$ and we analyzed the performance of these codes. In this section we formalize the problem of code design within an appropriate algebraic framework.

The sign code is a rate k/n binary ($\mathbb{F}_2$) convolutional code with generator matrix $G = [g_{ij}(D)]$, where

$$g_{ij}(D) = g_{ij0} + g_{ij1}D + \cdots + g_{ijl(ij)}D^{l(ij)}$$

and $g_{ijk} = 0$ or 1. A coset of this code generates the channel inputs. We fix n binary sequences a^i, $i = 0,1,\cdots,n-1$. Then k binary ($\mathbb{F}_2$) messages $y^i = (y^i_0, y^i_1, \cdots)$ determine n output sequences z^i, $i = 0,1,\cdots,n-1$, according to the rule

$$[z^0,\cdots,z^{n-1}] = [y^0,\cdots,y^{k-1}]G + [a^0,\cdots,a^{n-1}].$$

The ± 1 valued channel input x is given by

$$x_{jn+t} = (-1)^{z^t_j}.$$

It is important to choose a sign code that is noncatastrophic. Otherwise, for every set of inputs y^i, there is a second set of inputs $\hat{y}^i$, and corresponding output sequences $\hat{z}^i$, for which

$$z^i \approx \hat{z}^i \qquad i = 0,1,\cdots,n-1.$$

If the channel inputs agree in all but finitely many positions then there is no hope of distinguishing the channel outputs. Example 4 in Section II is a rate 1/2 code for the $(1-D^2)/2$ channel. There are input sequences y for which there exists an input sequence $\hat{y}$, and corresponding output $\hat{z}$, with $z \approx \hat{z}$. However as $M \to \infty$ the fraction of input sequences of length M with this property tends to zero. If the sign code is catastrophic then this fraction is always 1.

The next lemma plays a role in the method of identifying all codewords x, x' for which $x \not\approx x'$ and $(1 - D^N)x/2 \approx (1 - D^N)x'/2$.

Lemma 1: Let G be the generator matrix of a rate k/n sign code. Let m^i, w^i, $i = 0,1,\cdots,k-1$ be ± 1 valued message sequences, let x, x' be the corresponding ± 1 valued channel inputs, and let

$$m^i_j = (-1)^{y^i_j} w^i_j \qquad i = 0,1,\cdots,k-1.$$

where $y^i_j = 0$ or 1. If

$$[y^0,\cdots,y^{k-1}]G = [z^0,\cdots,z^{n-1}],$$

then for all j

$$x_{jn+t} = (-1)^{z^t_j} x'_{jn+t} \qquad t = 0,1,\cdots,n-1,$$

and this property is independent of the coset of the sign code used to generate the channel inputs.

Example: In Example 3, we showed that codewords x, x' for which $x \not\approx x'$, and $(1-D)x/2 \approx (1-D)x'/2 \pmod 2$ correspond to messages $[m^0, m^1]$ and $[-m^0, m^1]$. Applying Lemma 1 with $y^0 \approx 1/(1+D)$, $y^1 \approx \mathbf{0}$,

$$G = \begin{bmatrix} 1 & D & 1 \\ 0 & 1 & D \end{bmatrix},$$

we see that $x' = -x$, and so $(1-D)x/2 \approx \mathbf{0}$.

The *magnitude code* is the rate k/n convolutional code that results from using the sign code to generate inputs to an $\mathbb{F}_2$ channel with transfer function $1 + D^N$. Let δ be the $n \times n$ matrix

$$\delta = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & \ddots & 1 \\ D & & & & 0 \end{bmatrix}. \tag{1}$$

If F is the generator matrix of the magnitude code then $F = G(I + \delta^N)$. The matrix $I + \delta^N$ is called the channel matrix and it represents the operation of adding a sequence of n-bit bytes to an N-bit shift of itself (see Section II for examples). The next lemma is used to identify all codewords x, x' such that $x \not\approx x'$ and $(1 - D^N)x/2 \approx (1 - D^N)x'/2 \pmod 2$.

Lemma 2: Let F be the generator matrix of a rate k/n magnitude code. Let $m^i, w^i, i = 0, 1, \cdots, k-1$ be ± 1 valued message sequences, let x, x' be the corresponding ± 1 valued channel inputs, and let

$$m_j^i = (-1)^{y_j^i} w_j^i \qquad i = 0, 1, \cdots, k-1,$$

where $y_j^i = 0$ or 1. If

$$[y^0, \cdots, y^{k-1}]F = [z^0, \cdots, z^{n-1}],$$

then for all j,

$$\left|\frac{1}{2}(1 - D^N)(x - x')_{jn+t}\right|^2 = \begin{cases} 1, & \text{if } z_j^t = 1 \\ 0 \text{ or } 4, & \text{if } z_j^t = 0 \end{cases} \qquad t = 0, 1, \cdots, n-1.$$

This property is independent of the coset of the sign code used to generate the channel inputs.

Example: The rate 1/2 magnitude code $F = [1 + D^2 + D^3 + D^4, 1 + D + D^2 + D^4]$ results from using the sign code $G = [1 + D + D^3, 1 + D^2 + D^3]$ to generate inputs to an $\mathbb{F}_2$ channel with transfer function $1 + D^2$. The equation

$$\left[\frac{1}{1 + D}\right] F \approx [0, 0]$$

says that the outputs of the $(1 - D^2)/2$ channel corresponding to ± 1 valued inputs $m, -m$, agree modulo 2 in all but finitely many places.

The minimum squared Euclidean distance

$$d^2 = \min_{x \neq x^*} \left\{ \left\| \frac{1}{2}(1 - D^N)(x - x^*) \right\|^2 \right\}$$

of the recording code is bounded below by the free distance of the magnitude code. Example 5 in Section II shows that this lower bound is not always tight.

The probability of decoder error is determined by the minimum squared Euclidean distance d^2. The problem of code design is to maximize d^2 while keeping the number of decoder states as small as possible. A procedure for constructing codes is described in Section IV.

The examples described in Section II show that two further problems can occur. The first problem occurs when the magnitude code is catastrophic. We must then identify pairs of ± 1 valued messages m^i, m'^i, $i = 0, 1, \cdots, k-1$, such that $m^j \not\approx m'^j$ for some j, and such that the corresponding codewords x, x' satisfy $(1 - D^N)x/2 \approx (1 - D^N)x'/2$. We shall say that an output sequence $(1 - D^N)x/2$ with the above property is flawed. A code is flawed if there exist flawed output sequences and a code is catastrophic if every output sequence is flawed. The examples described in Section II show that it is sometimes possible to eliminate flaws by generating channel inputs using an appropriate coset of the sign code. The identification of all flawed output sequences is carried out using Lemmas 1 and 2. First we use Lemma 2 to find all pairs of messages m^i, m'^i such that $m'^j \not\approx m^j$ for some j, and such that the corresponding codewords x, x' satisfy $(1 - D^N)x/2 \approx (1 - D^N)x'/2 \pmod 2$. Then we apply Lemma 1 to identify the flawed output sequences.

The second problem is to eliminate long sequences of zeros at the output by generating channel inputs using an appropriate coset of the sign code. It is necessary to choose a nontrivial coset. Otherwise the channel output corresponding to message $m^i = (1, 1, 1, \cdots)$ is zero in all but finitely many positions. In Section V we solve the problem of zero-run length limiting the output sequence.

IV. Code Construction

This section discusses methods of constructing recording codes with large squared Euclidean distance d^2. (Of course the most reliable method is exhaustive search of the class of noncatastrophic sign codes.) The papers by Forney [14, 15] on the algebraic structure of convolutional codes are important and useful references.

The construction procedure is as follows (write $n = sb$, $N = sc$, where $gcd(b, c) = 1$).

Step 1) Choose a $k \times n$ generator matrix G_0 for a binary convolutional code with a feedback-free delay-free inverse G_0^{-1}. (The entries of G_0, G_0^{-1} are polynomials and $G_0 G_0^{-1} = I_k$; G_0 is a *basic encoder*.)

Step 2) Form $M = G_0(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})$, and obtain the invariant factor decomposition (or Smith normal form) of M (see [14] for an algorithm). We write $M = A\,\mathrm{diag}[\gamma_1, \cdots, \gamma_k]M^*$ where $\gamma_1 | \gamma_2 | \cdots | \gamma_k$ are the invariant factors of M, the $k \times k$ matrix A is unimodular (polynomial entries and determinant 1), and M^* is a $k \times n$ matrix with a feedback-free delay-free polynomial inverse.

The invariant factors $\gamma_1, \cdots, \gamma_k$ are unique since $\gamma_i = \Delta_i / \Delta_{i-1}$, where Δ_i is the greatest common divisor of the $i \times i$ minors of M. The matrices A and M^* are not in general unique.

Step 3) Choose a $k \times k$ unimodular matrix B (polynomial entries and determinant 1) which minimizes the constraint length of the code generated by $BM^*(I + \delta^{sc})$ (see [15] for an algorithm). Then set

$$G = BM^* \text{ and } F = G(I + \delta^{sc}).$$

Observe that the polynomial matrix G has a feedback-free delay-free polynomial inverse and that

$$\begin{aligned} F &= B\,\mathrm{diag}[1/\gamma_1, \cdots, 1/\gamma_k]A^{-1} \\ &\quad \cdot G_0(I + \delta^{sc} + \cdots + \delta^{sc(b-1)})(I + \delta^{sc}) \\ &= (1 + D^c)B\,\mathrm{diag}[1/\gamma_1, \cdots, 1/\gamma_k]A^{-1}G_0 \\ &= RG_0. \end{aligned}$$

The entries of the matrix $R = FG_0^{-1}$ are polynomials.

Theorem 3: The free distance of F is at least the free distance of the original convolutional code G_0.

Proof: Let $y(D)$ be a polynomial input that minimizes the weight of $y(D)F$. Since $y(D)F = (y(D)R)G_0$, and since $y(D)R$ is a polynomial input, this weight is bounded below by the free distance of G_0.

The codes constructed by this procedure are not catastrophic but they may be flawed as the examples below illustrate.

Example 6: Here $N = 2, n = 2, k = 1, s = 2, b = 1, c = 1$. The convolutional code $G_0 = [1 + D + D^3, 1 + D^2 + D^3]$ has free distance 6 and constraint length 3. Now $I + \delta^{sc} + \cdots + \delta^{sc(b-1)} = I$ so $M = M^* = G = G_0$ and $F = [1 + D^2 + D^3 + D^4, 1 + D + D^2 + D^4]$. The free distance of the magnitude code F is 8 and the constraint length is 4.

Next we find all flawed output sequences. Consider codewords $\boldsymbol{x}, \boldsymbol{x}'$, corresponding to messages $\boldsymbol{m}, \boldsymbol{m}'$, such that $\boldsymbol{x} \not\approx \boldsymbol{x}'$ and $(1 - D^2)\boldsymbol{x}/2 \approx (1 - D^2)\boldsymbol{x}'/2$. Write $m_j = (-1)^{y_j} m_j'$, where $y_j = 0$ or 1. Then by Lemma 2

$$\frac{1}{2}(1 - D^2)\boldsymbol{x} \approx \frac{1}{2}(1 - D^2)\boldsymbol{x}' \pmod 2 \Leftrightarrow yF \approx [\mathbf{0}, \mathbf{0}].$$

thus $y \approx 1/(1 + D)$, and Lemma 1 implies $\boldsymbol{x}' = -\boldsymbol{x}$. The only flawed output sequence is the all-zero sequence.

Example 7: Here $N = 2, n = 3, k = 2, s = 1, b = 3, c = 2$. The convolutional code

$$G_0 = \begin{bmatrix} D & D & 1 + D \\ 1 + D + D^2 & 1 + D^2 & D \end{bmatrix}$$

has free distance 4 and constraint length 3. We have

$$I + \delta^{sc} + \delta^{2sc} = \begin{bmatrix} 1 & D & 1 \\ D & 1 & D \\ D^2 & D & 1 \end{bmatrix},$$

$$M = G_0(I + \delta^{sc} + \delta^{2sc}) = \begin{bmatrix} D + D^3 & 0 & 1 + D^2 \\ 1 + D^2 & 1 + D + D^2 + D^3 & 1 + D + D^2 + D^3 \end{bmatrix}.$$

The invariant factors of the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & D \end{bmatrix} G_0(I + \delta^{sc} + \delta^{2sc}) = \begin{bmatrix} 1 + D^2 & 1 + D + D^2 + D^3 & 1 + D + D^2 + D^3 \\ 0 & D + D^2 + D^3 + D^4 & 1 + D + D^3 + D^4 \end{bmatrix}$$

are $\gamma_1 = \gamma_2 = 1 + D^2$ so that

$$M^* = \frac{1}{1 + D^2}\begin{bmatrix} 0 & 1 \\ 1 & D \end{bmatrix} G_0(I + \delta^{sc} + \delta^{2sc})$$
$$= \begin{bmatrix} 1 & 1 + D & 1 + D \\ 0 & D + D^2 & 1 + D + D^2 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} D & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$G = BM^* = \begin{bmatrix} D & 0 & 1 \\ 1 & 1 + D & 1 + D \end{bmatrix},$$

and

$$F = G\begin{bmatrix} 1 & 0 & 1 \\ D & 1 & 0 \\ 0 & D & 1 \end{bmatrix} = \begin{bmatrix} D & D & 1 + D \\ 1 + D + D^2 & 1 + D^2 & D \end{bmatrix}.$$

Here the magnitude code F is the original convolutional code. There are no flawed output sequences because Lemma 2 implies that there are no pairs of codewords $\boldsymbol{x}, \boldsymbol{x}'$ with $\boldsymbol{x} \not\approx \boldsymbol{x}'$ and $(1 - D)\boldsymbol{x}/2 \approx (1 - D)\boldsymbol{x}'/2 \pmod 2$.

Example 8: Here $N = 1, n = 4, k = 3, s = 1, b = 4, c = 1$. The convolutional code

$$G_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 + D & D & 0 & 1 \\ 0 & D & 1 + D & 1 + D \end{bmatrix}$$

has free distance 3 and constraint length 2. We have

$$I + \delta^{sc} + \delta^{2sc} + \delta^{3sc} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ D & 1 & 1 & 1 \\ D & D & 1 & 1 \\ D & D & D & 1 \end{bmatrix},$$

$$M = G_0(I + \delta^{sc} + \delta^{2sc} + \delta^{3sc})$$
$$= \begin{bmatrix} 1+D & 0 & 1+D & 0 \\ 1+D^2 & 1+D & 1+D & 0 \\ D^2 & D & 1+D+D^2 & D \end{bmatrix}.$$

The invariant factors of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 + D & 1 + D \\ D & D & 1 + D \end{bmatrix} G_0(I + \delta^{sc} + \delta^{2sc} + \delta^{3sc})$$
$$= \begin{bmatrix} 1 & 1 & D^2 & D \\ 0 & 1 + D & 1 + D + D^2 + D^3 & D + D^2 \\ 0 & 0 & 1 + D^3 & D + D^2 \end{bmatrix}$$

are $\gamma_1 = 1, \gamma_2 = 1 + D, \gamma_3 = 1 + D$, so that

$$M^* = \operatorname{diag}[1, 1/(1 + D), 1/(1 + D)]$$
$$\cdot \begin{bmatrix} 1 & 1 & D^2 & D \\ 0 & 1 + D & 1 + D + D^2 + D^3 & D + D^2 \\ 0 & 0 & 1 + D^3 & D + D^2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & D^2 & D \\ 0 & 1 & 1 + D^2 & D \\ 0 & 0 & 1 + D + D^2 & D \end{bmatrix}.$$

Choose

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & D & 1+D \end{bmatrix}.$$

Then

$$G = BM^* = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & D & 0 \\ 0 & D & 1+D & D \end{bmatrix}.$$

and

$$F = G \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ D & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1+D & D \\ D^2 & D & 1 & 1 \end{bmatrix}.$$

The free distance of the magnitude code is 4 and the constraint length is 3.

Again we find all flawed output sequences. Consider codewords x, x', corresponding to messages m^i, m'^i, $i = 0,1,2$, such that $x \not\approx x'$ and $(1-D)x/2 \approx (1-D)x'/2$. Write $m_j^i = (-1)^{y_j^i} m_j'^i$, where $y_j^i = 0$ or 1. Then by Lemma 2

$$\frac{1}{2}(1-D)x \approx \frac{1}{2}(1-D)x' (\mathrm{mod}\, 2)$$

$$\Leftrightarrow [y^0, y^1, y^2] F = [\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}],$$

and so $y^0 \approx 1/(1+D)$, $y^1 \approx \mathbf{0}$, and $y^2 \approx 1/(1+D)$. Now Lemma 1 implies $x' = -x$ and so the only flawed output sequence is the all-zero sequence.

Let us briefly consider an alternative construction method. Write $n = sb$, $N = sc$, where $gcd(b,c) = 1$. Suppose we know that F generates a rate k/n convolutional code with free distance d. If there exists a matrix G for which

$$G(I + \delta^{sc}) = F,$$

then F is the magnitude code corresponding to the sign code G. Since

$$(I + \delta^{sc})(I + \delta^{sc} + \cdots + \delta^{sc(b-1)}) = (1 + D^c)I,$$

we can always solve the equation

$$G(I + \delta^{sc}) = (1 + D^c)F.$$

Note that the free distance of $(1 + D^c)F$ is even, and that multiplication by $(1 + D^c)$ increases the number of states required by the decoder by a factor of 2^c.

Wolf and Ungerboeck apply the following procedure to find binary convolutional codes for the $1 - D$ channel in [11]. First they choose a generator matrix G_0 for a good binary convolutional code ("good" means large Hamming distance and small constraint length). The encoder (sign code) is then taken to be the matrix $G = G_0(I - \delta)^{-1}$, so the codewords generated by G_0 are passed through the inverse of the channel matrix before they are introduced to the channel. The generalization to the $(1 - D^N)$ channel is to multiply G_0 by $(I - \delta^{sc})^{-1} = (1 - D^c)^{-1}(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})$. Wolf and Ungerboeck refer to this inversion procedure as precoding (the procedure occurs before the channel) and this is standard terminology in the magnetic recording area. Perhaps a more descriptive name is "postcoding" since the procedure takes place after the encoder. The magnitude code $F = G_0$, and the Euclidean free distance is bounded below by the Hamming distance of G_0. Wolf and Ungerboeck prove that this lower bound is achieved when the free distance of G_0 is even. They also prove that when the free distance of G_0 is odd the lower bound is one less than the true distance. They also prove (again for the $1 - D$ channel) that the constraint length of the code is bounded above by the constraint length of G_0 plus 1.

In our approach the free distance and the constraint length of the overall code is just a function of the magnitude code. In the approach taken by Wolf and Ungerboeck [11] the relationship between the sign and magnitude codes must be studied to determine the distance and complexity.

The use of a precoded generator will generally imply that the generator will have rational terms; the encoding function is recursive (i.e., not freeback-free). The advantages and disadvantages of such encoders are described by Forney [14]. Forney proves that every binary convolutional code has a systematic, nonfeedback-free encoder. Since the method of Wolf and Ungerboeck involves encoders with feedback, they rightfully chose the original matrix G_0 to be systematic; this has the possible advantage that the message appears as a subsequence of the magnitude codeword. One possible disadvantage of such encoders, as noted by Forney, is that the average number of data errors associated with the most likely error events tends to be larger with such encoders. This is probably not a serious drawback in magnetic recording systems since the frequency of the error events is usually a more critical parameter of the system than the average bit error rate.

V. CLOCK SYNCHRONIZATION

A. Choosing the Coset

Clock synchronization is maintained by eliminating long sequences of zeros as possible channel outputs. This is accomplished by choosing an appropriate coset of the sign code to generate the channel inputs.

We consider a rate k/n sign code with generator matrix G and a coset a^i, $i = 0,1,\cdots,n-1$. We restrict our attention to $\mathbb{F}_2$-sequences a^i that are periodic and we are particularly interested in sequences for which this period is small. If the greatest common divisor (gcd) of the periods of the sequences a^i is p then the decoder requires a trellis that is time-varying with period p. If $a^i = a^i(D) = 1/(1+D)$ or 0 then the recording code is said to be *stationary*. Stationary recording codes should be used whenever possible.

Messages $y^i = (y_0^i, y_1^i, \cdots)$, $y_j^i = 0$ or 1, determine n output sequences z^i, $i = 0,1,\cdots,n-1$, according to the rule

$$[z^0, \cdots, z^{n-1}] = [y^0, y^1, \cdots, y^{k-1}]G + [a^0, a^1, \cdots, a^{n-1}],$$

and the ± 1 valued channel input $\boldsymbol{x}$ is given by

$$x_{jn+t} = (-1)^{z_j^t}.$$

The codeword $\boldsymbol{x}$ is input to the $(1 - D^N)/2$ channel. The zero sequence is a possible channel output if and only if

$$\left([y_0, \boldsymbol{y}^1, \cdots, \boldsymbol{y}^{k-1}]G + [\boldsymbol{a}^0, \cdots, \boldsymbol{a}^{n-1}]\right)(I + \delta^N) \approx [\boldsymbol{0}, \cdots, \boldsymbol{0}],$$

where δ is the matrix defined in (1). Thus

$$[\boldsymbol{y}^0, \boldsymbol{y}^1, \cdots, \boldsymbol{y}^{k-1}]F \approx [\boldsymbol{a}^0, \cdots, \boldsymbol{a}^{n-1}](I + \delta^N),$$

where $F = G(I + \delta^N)$ is the generator matrix of the magnitude code. The problem of zero-run length limiting the output sequence is precisely that of choosing $\mathbb{F}_2$-sequences $\boldsymbol{a}^i$, $i = 0, 1, \cdots, n-1$, so that $[\boldsymbol{a}^0, \cdots, \boldsymbol{a}^{n-1}](I + \delta^N)$ does not agree with a codeword of the magnitude code in all but finitely many positions. We begin this section by showing how to choose appropriate sequences $\boldsymbol{a}^i$, $i = 0, 1, \cdots, n-1$.

Write $n = sb$, $N = sc$, where $gcd(b, c) = 1$. We may suppose that the $k \times n$ generator matrix G is a basic encoder for the sign code. Thus the *gcd* of the $k \times k$ minors of G equals 1. Define

$\Delta = gcd$ of the $k \times k$ minors of the generator matrix

$F = G(I + \delta^N)$ for the magnitude code.

We order the ways to choose a k-subset of an n-set and we use this ordering to arrange $k \times k$ minors in matrix form. Let $G^{(k)} = (g_1, \cdots, g_L)$, $L = \binom{n}{k}$, be the vector of $k \times k$ minors of G, and let $(I + \delta^N)^{(k)}$ denote the kth compound matrix of $I + \delta^N$. The ijth entry of the $L \times L$ matrix $(I + \delta^N)^{(k)}$ is the ijth $k \times k$ minor of $I + \delta^N$. The entries f_i, $i = 1, \cdots, L$, of the vector $F^{(k)} = G^{(k)}(I + \delta^N)^{(k)}$ are the $k \times k$ minors of F. This is just the Binet–Cauchy theorem (see for example Gantmacher [16, p. 9, vol. 1]). Thus $\Delta = gcd(f_1, \cdots, f_L)$ and we say that Δ is the *content* of $F^{(k)}$.

Lemma 4: Δ divides $\det(I + \delta^N)$.

Proof: The ordering of k-sets induces a natural ordering of $(n-k)$-sets; the ith $(n-k)$-set is just the complement of the ith k-set. Define an $L \times L$ matrix $(I + \delta^N)^{(n-k)}$, where the ijth entry is the ijth $(n-k) \times (n-k)$ minor of $I + \delta^N$. The identity

$$(I + \delta^N)^{(k)}\left[(I + \delta^N)^{(n-k)}\right]^T = \det(I + \delta^N)I$$

is a generalization of the usual expansion of the determinant in terms of 1×1 minors and $(n-1) \times (n-1)$ minors, and can be found in Gantmacher [16] or Aitken [17]. Now

$$F^{(k)}\left[(I + \delta^N)^{(n-k)}\right]^T = \det(I + \delta^N)G^{(k)},$$

so Δ divides the content of $\det(I + \delta^N)G^{(k)}$, which is just $\det(I + \delta^N)$.

Lemma 5: $\det(I + \delta^N) = (1 + D^c)^s$.

Proof: If $s = gcd(n, N) = 1$, then the permutation $a \to a + N \pmod n$ is a single cycle and the matrix $I + \delta^N$ is irreducible. In general we may reorder rows and columns so that $I + \delta^N$ is block diagonal and there are s identical irreducible diagonal blocks. The diagonal block is the $b \times b$ matrix $I + \delta^c$. Therefore it is sufficient to prove the lemma for the case $s = 1$.

There is a unique way to write $I + \delta^N$ as a sum

$$I + \delta^N = D_1P_1 + D_2P_2,$$

where D_1, D_2 are diagonal matrices and P_1, P_2 are permutation matrices. The permutation P_1 is the identity and the permutation P_2 is $a \to a + N \pmod n$. The matrix $D_1 = I$ and $D_2 = \mathrm{diag}[1, \cdots, 1, D, \cdots, D]$, where the number of D's is N. Therefore

$$\det(I + \delta^N) = \det(D_1) + \det(D_2) = 1 + D^N.$$

Example: A class of rate 3/4 sign codes $G = [g_{ij}(D)]$ for the $1 - D^2$ channel. Thus $N = 2$, $n = 4$, $s = 2$, $b = 2$, $c = 1$. We order g_1, g_2, g_3, g_4, the 3×3 minors of G, by taking g_i to be that minor obtained by suppressing the ith column of G. Then we have

$$I + \delta^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ D & 0 & 1 & 0 \\ 0 & D & 0 & 1 \end{bmatrix}$$

$$(I + \delta^2)^{(3)} = \begin{bmatrix} 1+D & 0 & D(1+D) & 0 \\ 0 & 1+D & 0 & D(1+D) \\ 1+D & 0 & 1+D & 0 \\ 0 & 1+D & 0 & 1+D \end{bmatrix}.$$

Thus

$$F^{(3)} = (f_1, f_2, f_3, f_4) = (1 + D) \cdot (g_1 + g_3, g_2 + g_4, D_{g1} + g_3 Dg_2 + g_4).$$

Clearly $\Delta = gcd(f_1, f_2, f_3, f_4) = 1 + D$ or $(1 + D)^2$, and

$$\Delta = (1 + D)^2 \Leftrightarrow g_1 \equiv g_3 \bmod (1 + D) \text{ and } g_2 \equiv g_4 \bmod (1 + D).$$

Since G is a basic encoder, the row space $R(G)$ modulo $1 + D$ is three-dimensional over $\mathbb{F}_2$ and so $R(G) = \boldsymbol{w}^\perp \bmod (1 + D)$ for some binary vector $\boldsymbol{w}$ of length 4. It is straightforward to prove that

$$\Delta = (1 + D)^2 \Leftrightarrow \boldsymbol{w} = (1,1,1,1), (0,1,0,1), \text{ or } (1,0,1,0).$$

Next we consider the problem of finding a basic syndrome former for the magnitude code $F = G(I + \delta^{sc})$. The identity $C^{(k)}[C^{(n-k)}]^T = (\det C)I$ used to prove Lemma 4 will be applied again.

We write the generator matrix G for the sign code in the form

$$G = A[I_k, 0]B,$$

where A and B are unimodular matrices. Then

$$H = [0, I_{n-k}](B^{-1})^T$$

is a basic syndrome former.

Lemma 6: The *gcd* Δ^* of the $(n-k)\times(n-k)$ minors of the matrix

$$E = H(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})^T$$

is given by

$$\Delta^* = \Delta(1 + D^c)^{n-k-s},$$

where Δ is the *gcd* of the $k \times k$ minors of the generator matrix $F = G(I + \delta^{sc})$ of the magnitude code.

Proof: Given that

$$F = [I_k, 0]B(I + \delta^{sc}),$$

we have

$$F^{(k)} = (1, 0, \cdots, 0)B^{(k)}(I + \delta^{sc})^{(k)}.$$

$$(I + \delta^{sc})(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)}) = (1 + D^c)I,$$

and so

$$I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)} = (1 + D^c)(I + \delta^{sc})^{-1}.$$

Thus

$$E = [0, I_{n-k}](B^{-1})^T\left[(1 + D^c)(I + \delta^{sc})^{-1}\right]^T,$$

and

$$E^{(n-k)} = (1, 0, \cdots, 0)\left[(B^{(n-k)})^T\right]^{-1} \cdot (1 + D^c)^{n-k}\left[\left((I + \delta^{sc})^{(n-k)}\right)^T\right]^{-1}.$$

(Recall from Lemma 4 that the ijth entry of $E^{(n-k)}$ is the ijth $(n-k)\times(n-k)$ minor of E and that the orderings of $k \times k$ and $(n-k)\times(n-k)$ minors are complementary.) The identities

$$B^{(k)}(B^{(n-k)})^T = (\det B)I = I,$$

$$(I+\delta^{sc})^{(k)}\left[(I+\delta^{sc})^{(n-k)}\right]^T = \left[\det(I+\delta^{sc})\right]I = (1+D^c)^s I,$$

give

$$E^{(n-k)} = (1 + D^c)^{n-k-s}(1, 0, \cdots, 0)B^{(k)}(I + \delta^{sc})^{(k)}.$$

The content Δ^* of $E^{(n-k)}$ differs from the content Δ of $F^{(k)}$ by the multiplicative factor $(1 + D^c)^{n-k-s}$.

Lemma 7: Let $d_1|d_2|\cdots|d_{n-k}$ be the invariant factors of $E = H(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})^T$. Then every invariant factor divides $1 + D^c$.

Proof: Write

$$E = A' \operatorname{diag}[d_1, \cdots, d_{n-k}]E',$$

where A' is a unimodular $k \times k$ matrix and E' is a $k \times n$ polynomial matrix with a feedback-free, delay-free polynomial inverse E'^{-1}. Since

$$FE^T = G(1 + D^c)IH^T = 0,$$

the polynomial matrix E' is a basic syndrome former for the magnitude code $F = G(I + \delta^{sc})$. Since the matrix

$$E'(I + \delta^{sc})^T = \operatorname{diag}\left[\frac{1}{d_1}, \cdots, \frac{1}{d_{n-k}}\right](1 + D^c)H$$

has polynomial entries, and since the *gcd* of the entries in each row of H equals 1, we may conclude that every invariant factor d_i divides $(1 + D^c)$.

Theorem 8: Let $G = A[I_k, 0]B$ be a basic encoder for the sign code and let $H = [0, I_{n-k}](B^{-1})^T$ be a basic syndrome former. Let $d_1|d_2|\cdots|d_{n-k}$ be the invariant factors of $E = H(I + \delta^{sc} + \cdots + \delta^{sc(b-1)})^T$, and let $\lambda(D) = (1 + D^c)/d_{n-k}$.

Then the sign code G can be made zero-run length limiting by generating channel inputs using a coset

$$a(D) = \frac{1}{1 + D^P}\left(a^0(D), \cdots, a^{n-1}(D)\right),$$

where $a^j(D) = 1$ for some j, $a^i(D) = 0$ for $i \neq j$, if and only if $1 + D^P \nmid \lambda(D)$.

Proof: The zero-run length is limited precisely when the sequence $a(D)(I + \delta^{sc})$ does not agree with a codeword of the magnitude code in all but finitely many positions. If $1 + D^P|\lambda(D)$ then

$$a(D)(I + \delta^{sc})E'^T = (1 + D^c)a(D)H^T \operatorname{diag}\left[\frac{1}{d_1}, \cdots, \frac{1}{d_{n-k}}\right].$$

Since $1 + D^P|(1 + D^c)/d_i$ for all i, every entry in this vector of length $n-k$ is a polynomial. Hence $a(D)(I + \delta^{sc})$ agrees with some codeword of the magnitude code in all but finitely many positions.

If $1 + D^P \nmid \lambda(D)$ then there exists an entry $h_{(n-k)j}$ in the last row of H for which

$$\frac{(1 + D^c)h_{(n-k)j}}{(1 + D^P)d_{n-k}} = \frac{\lambda(D)h_{(n-k)j}}{1 + D^P}$$

is not a polynomial. If $a(D) = [1/(1 + D)]$ $(0, \cdots, 0, 1, 0, \cdots, 0)$, where the single nonzero entry is in position j, then $a(D)(I + \delta^{sc})E'^T$ is not a vector with polynomial entries. Hence $a(D)(I + \delta^{sc})$ does not agree with a codeword of the magnitude code in all but finitely many positions. The sign code can be made zero-run length limiting by periodically changing the sign of a single channel input.

We specialize Theorem 8 to the case $n = k + 1$ and obtain the following corollary.

Corollary 9: Let G be a rate $k/(k+1)$ sign code and let Δ be the *gcd* of the $k \times k$ minors of the magnitude code $F = G(I + \delta^{sc})$. Then G can be made zero-run length limiting by generating channel inputs using a coset

$$a(D) = \frac{1}{1 + D^P}\left(a^0(D), \cdots, a^{n-1}(D)\right),$$

where the degree of the polynomial $a^i(D)$ is less than P, if and only if

$$1 + D^P \nmid \frac{(1 + D^c)^s}{\Delta}.$$

Proof: By Lemma 6

$$d_{n-k} = d_1 = \Delta(1 + D^c)^{n-k-s} = \Delta(1 + D^c)^{1-s}.$$

Example 9: Again we consider rate 3/4 sign codes G for the $(1-D^2)/2$ channel; $N=2, n=4, s=2, b=2, c=1$. By Lemma 6

$$\Delta^* = \Delta(1+D)^{-1},$$

and since Δ^* divides $1+D$ we have

$$\Delta = 1 + D \text{ or } 1 + D^2.$$

If $\Delta = 1 + D^2$, then it is possible to zero-run length limit G by changing the sign of a single channel input. If $\Delta = 1 + D$, then a stationary trellis is not possible. However a time-varying trellis with period 2 is possible.

Example 10: Consider rate 1/2 sign codes G for the $(1-D)/2$ channel; $N=1, n=2, s=1, b=2, c=1$. By Lemma 6, $\Delta^* = \Delta$, and since Δ^* divides $(1+D)$ we have $\Delta = 1$ or $1+D$. Corollary 9 shows that it is possible to zero-run length limit G using a stationary trellis if and only if $\Delta = 1 + D$. Since G is noncatastrophic, this is precisely the case when the entries $g_{11}(D)$, $g_{12}(D)$ satisfy $g_{11}(1) = g_{12}(1) = 1$.

Example 11: Consider rate 1/2 sign codes $G = [g_{11}(D), g_{12}(D)]$ for the $(1-D^2)/2$ channel; $N=2, n=2, s=2, b=1, c=1$. It is clear that $\Delta = (1+D)$. Since $(1+D^c)^s/\Delta = 1 + D$, it is not possible to zero-run length limit G using a stationary trellis (of course this is easy to see directly). However a time-varying trellis with period 2 is possible.

B. Calculation of the Maximum Zero-Run Length

We have shown how to eliminate arbitrarily long sequences of zeros as channel outputs by choosing an appropriate coset of the sign code to generate channel inputs. In this subsection we assume that an appropriate coset has been chosen and we consider the problem of finding the maximum zero-run length. Baumert, McEliece, and van Tilborg [18] considered the problem of symbol synchronization in convolutionally coded systems, and we shall follow their approach here.

Let $G = A[I_k, 0]B$ be a basic encoder for the sign code, let $H = [0, I_{n-k}](B^{-1})^T$, and let $E = H(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})^T$. If $E = A'\operatorname{diag}[d_1, \cdots, d_{n-k}]E'$, where $d_1|d_2|\cdots|d_{n-k}$ are the invariant factors of E, A' is unimodular, and E' has a polynomial inverse $(E')^{-1}$, then

$$E' = \operatorname{diag}\left[\frac{1}{d_1}, \ldots, \frac{1}{d_{n-k}}\right](A')^{-1} \cdot H(I + \delta^{sc} + \cdots + \delta^{sc(b-1)})^T$$

is a basic syndrome former for the magnitude code $F = G(I + \delta^{sc})$.

If there are nontrivial invariant factors d_i, then there exist messages $m^i, m'^i, i = 0, 1, \cdots, k-1$, for which the corresponding codewords x, x' satisfy $x \not\approx x'$ and $(1-D^N)x/2 \approx (1-D^N)x'/2 \pmod 2$. If $(1-D^N)x/2 \approx (1-D^N)x'/2$, then the output sequence $(1-D^N)x/2$ is flawed. Sometimes it is possible to correct flaws by choosing an appropriate coset of the sign code (Example 3) and sometimes it is not possible (Example 4). The problem of distinguishing these two cases is addressed in Section VI. The problems presented by noncorrectable flaws can be finessed by observing that the fraction of messages of length M corresponding to flawed output sequences tends to 0 as M tends to ∞. The input to the encoder can then be constrained to avoid bad messages at some marginal loss in data rate. However it may be quite complicated to do this in practice.

However, magnitude codes with nontrivial invariant factors d_i have the following advantages.

1) As the degree of the polynomial $\lambda(D) = (1 + D^c)/d_{n-k}$ decreases so does the degree of the smallest polynomial $1 + D^p$ for which $1 + D^p \nmid \lambda(D)$. It is easier to zero-run length limit sign codes for which d_{n-k} is large.
2) Dividing the entries of the rows of the matrix $E = H(I + \delta^{sc} + \delta^{2sc} + \cdots + \delta^{sc(b-1)})^T$ by the invariant factors d_i reduces the memory required by the syndrome former. This will reduce the maximum zero-run length.

If E' is a basic syndrome former for the magnitude code, then we may write

$$E' = E_{s-1}D^{s-1} + \cdots + E_1 D + E_0,$$

where E_i is an $(n-k) \times n$ matrix with all entries 0 or 1. We form the $(n-k) \times sn$ matrix

$$V = [E_{s-1}E_{s-2} \cdots E_0].$$

Then a binary vector c of length sn can be extended (in both directions) to a codeword in the magnitude code if and only if $cv^T = 0$. Suppose we have shown that the sequence

$$q(D) = \frac{1}{1+D^p}(a^0(D), \cdots, a^{n-1}(D))(I + \delta^{sc})$$

does not agree with a codeword of the magnitude code in all but finitely many positions. We require the maximum length of a subsequence that agrees with a subsequence of a codeword of the magnitude code. The sequence $q(D)$ is periodic with period pn so there are pn different positions where such an agreement can begin. For each starting point l we follow the sequence $q(D)$, collecting the most recent sn terms in a vector c. If at position $l + M_l - 1$, the inner product $cV^T \neq 0$ for the first time, then M_l is the maximum length of an agreement starting in position l. After pn calculations we know the maximum zero-run length.

Example: There are 128 rate 1/2 sign codes for the $(1-D)/2$ channel that require a decoder with no more than eight states. There are 12 noncatastrophic codes with squared Euclidean distance $d^2 = 6$ and they are listed in Table I. Consider the first entry of this table.

A basic syndrome former for the magnitude code is

$$[1 + D + D^2, 1 + D^2] = [1,1] + [1,0]D + [1,1]D^2,$$

and so

$$V = [11|10|11].$$

TABLE I
THE MAXIMUM ZERO-RUN LENGTH OF THE EIGHT STATE CODES FOR THE $(1-D)/2$ CHANNEL WITH SQUARED EUCLIDEAN DISTANCE $d^2 = 6$

Sign Code G	Magnitude Code F	Δ	Coset $a(D)$	Zero-Run Length
$1+D+D^3, D$	$1+D+D^2+D^3, 1+D^3$	$1+D$	$\frac{1}{1+D}[1,0]$	6
$1+D^2+D^3, D$	$1+D^3, 1+D+D^2+D^3$	$1+D$	$\frac{1}{1+D}[0,1]$	6
$D^2+D^3, 1$	$D+D^2+D^3, 1+D^2+D^3$	1	$\frac{1}{1+D^2}[1,0]$	7
$D^3, 1+D$	$D+D^2+D^3, 1+D+D^3$	1	$\frac{1}{1+D^2}[1,0]$	7
$1+D, D^2$	$1+D+D^3, 1+D+D^2$	1	$\frac{1}{1+D^2}[1,0]$	7
$1, D+D^2$	$1+D^2+D^3, 1+D+D^2$	1	$\frac{1}{1+D^2}[1,0]$	7
$1+D+D^2+D^3, 1$	$1+D^2+D^3, D+D^2+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8
$1+D^3, D$	$1+D^2+D^3, 1+D+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8
$1+D+D^2+D^3, D^2$	$1+D+D^2, 1+D+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8
$1+D^2+D^3, 1+D$	$1+D+D^3, D+D^2+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8
$1+D+D^2+D^3, D$	$1+D+D^3, 1+D^2+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8
$1+D+D^3, D+D^2$	$1+D+D^2, 1+D^2+D^3$	1	$\frac{1}{1+D^2}[1,0]$	8

The sequence

$$q(D) = \frac{1}{1+D}[1,0](I+\delta) = (1,1,1,\cdots).$$

An agreement starting in an even position can only extend over five positions since

$$(11|11|11)V^T \neq 0.$$

However an agreement starting in an odd position can extend over six positions ((01|11|11|10)).

If $\Delta = 1$, then there are no flawed output sequences. This is because Lemma 2 implies that there are no pairs of codewords x, x' with $x \not\approx x'$ and $(1-D)x/2 \approx (1-D)x'/2 \pmod 2$. If $\Delta = 1 + D$, Then Lemma 2 implies that if codewords x, x' satisfy $x \not\approx x', (1-D)x/2 \approx (1-D)x'/2$, then x, x' correspond to messages $\pm m$. Lemma 1 now implies $x' = -x$ and $(1-D)x/2 \approx 0$. Thus limiting the maximum zero-run length corrects all flaws.

VI. FLAW CORRECTION

Inputs to the $(1-D^N)/2$ channel are generated using a nontrivial coset $[a^0, a^1, \cdots, a^{n-1}]$ of a rate k/n sign code G. If the magnitude code $F = G(I + \delta^N)$ has nontrivial invariant factors, then there exist pairs of messages $m^i, m'^i, i = 0, 1, \cdots, k-1$, for which the corresponding codewords x, x' satisfy $x \not\approx x'$ and $(1-D^N)x/2 \approx (1-D^N)x'/2$. In this section we address the problem of eliminating flawed output sequences by an appropriate choice of the coset $[a^0, \cdots, a^{n-1}]$. Example 4 in Section II shows that this is not always possible. The main results of this section are theorems covering the most interesting cases; $N = 1, 2$ and $n = k+1$.

Let $n = k+1$, and let G be an encoder for the sign code with a feedback-free, delay-free inverse. Then the *gcd* of the $k \times k$ minors of G is 1. Write $n = sb$, $N = sc$ and suppose $N = s = c = 1$. The $1 \times (k+1)$ matrix $H = G^{(k)}$ is a basic syndrome former for G, and the $1 \times (k+1)$ matrix

$$\frac{F^{(k)}}{\Delta} = \frac{G^{(k)}(I + \delta + \delta^2 + \cdots + \delta^{n-1})^T}{\Delta}$$

is a basic syndrome former for the magnitude code, where Δ is the *gcd* of the $k \times k$ minors of $F = G(I+\delta)$. By Lemmas 4 and 5, we have $\Delta = 1$ or $1 + D$. Observe that if $G^{(k)} = (g_1, \cdots, g_{k+1})$, then

$$\begin{aligned}\Delta = 1 + D \Leftrightarrow\ & G^{(k)}(I + \delta + \delta^2 + \cdots + \delta^{n-1})^T \\ & \equiv 0 \bmod (1+D) \\ \Leftrightarrow\ & \sum_{i=1}^{k+1} g_i \equiv 0 \bmod (1+D).\end{aligned}$$

If $\Delta = 1$, then there are no flawed output sequences because Lemma 2 implies that there are no pairs of codewords x, x' with $x \not\approx x'$ and $(1-D)x/2 \approx (1-D)x'/2 \pmod 2$. Now suppose $\Delta = 1 + D$. Consider codewords x, x' corresponding to messages $m^i, m'^i, i = 0, 1, \cdots, k-1$, such that $x \neq x'$ and $(1-D)x/2 \approx (1-D)x'/2$. Write $m_j^i = (-1)^{y_j^i} m_j'^i$, where $y_j^i = 0$ or 1. Then by Lemma 2, $(1-D)x/2 \approx (1-D)x'/2 \pmod 2 \Leftrightarrow$

$[y^0,\cdots,y^{k-1}]G(I+\delta)\approx[\mathbf{0},\cdots,\mathbf{0}]$. Since G is a basic encoder

$$[y^0,\cdots,y^{k-1}]G\approx\frac{1}{1+D}[1,1,\cdots,1].$$

Lemma 1 now implies $x'=-x$, and so $(1-D)x/2\approx\mathbf{0}$. Thus flaw correction can be achieved by limiting the maximum zero-run length of possible output sequences. Theorem 10 below now follows directly from Corollary 9.

Theorem 10: Let G be a non-catastrophic rate $k/(k+1)$ sign code for the $(1-D)/2$ channel, let $F=G(I+\delta)$, and let Δ be the *gcd* of the $k\times k$ minors of F. Then

1) $\Delta=1$ or $1+D$;
2) $\Delta=1+D$, if and only if $G^{(k)}(1,1,\cdots,1)^T\equiv 0 \bmod (1+D)$;
3) if $\Delta=1$, then the sign code G is not flawed (nor is any coset). It is possible to limit the zero-run length by generating channel inputs using a coset $[a^0(D),\cdots,a^{n-1}(D)]$, where the sequences $a^i(D)$ have period 2. It is not possible to limit the zero-run length using a stationary trellis;
4) if $\Delta=1+D$, then there exist sequences $a^i(D)$, $i=0,1,\cdots,n-1$, with period 1 for which the coset $[a^0(D),\cdots,a^{n-1}(D)]$ of the sign code is not flawed and is zero-run length limited.

Remark: Theorem 10 implies that rate 1/2 sign codes $G=[g_{11}(D),g_{12}(D)]$ for which $g_{11}(1)=g_{12}(1)=1$ have significant advantages over other rate 1/2 sign codes for the $(1-D)/2$ channel (see also Table I).

Next we consider rate $k/(k+1)$ codes for the $(1-D^2)/2$ channel in the case where $k+1$ is even. Thus $s=2$, $c=1$. By Lemma 6 the $1\times(k+1)$ matrix

$$\frac{F^{(k)}}{\Delta}=\frac{(1+D)G^{(k)}(I+\delta^2+\cdots+\delta^{n-2})^T}{\Delta}$$

is a basic syndrome former for the magnitude code. By Lemma 7 every invariant factor of F divides $1+D$. The invariant factors of F are $1,1,\cdots,1,1+D$, or $1,\cdots,1,1+D,1+D$, since by Lemmas 4 and 5, the product of the invariant factors divides $1+D^2$. Note that by Lemma 6 we have $\Delta\neq 1$, since Δ^* is a polynomial and $(1+D^c)^{n-k-s}=1/(1+D)$.

Consider codewords x,x' corresponding to messages m^i,m'^i, $i=0,1,\cdots,k-1$, such that $x\neq x'$ and $(1-D^2)x/2\approx(1-D^2)x'/2$. Write $m^i_j=(-1)^{y^i_j}m'^i_j$. Then by Lemma 2,

$$\frac{1}{2}(1-D^2)x\approx\frac{1}{2}(1-D^2)x'(\bmod 2)$$
$$\Leftrightarrow [y^0,\cdots,y^{k-1}]G(I+\delta^2)\approx[\mathbf{0},\cdots,\mathbf{0}].$$

The binary sequences annihilated by $I+\delta^2$ are $0,1/(1+D),1/(1+D^2)$, and $D/(1+D^2)$. Thus

$$[y^0(D),\cdots,y^{k-1}(D)]G\approx\frac{1}{1+D}[1,1,\cdots,1],\quad \frac{1}{1+D}[1,0,1,0,\cdots,1,0],\quad \text{or }\frac{1}{1+D}[0,1,0,1,\cdots,0,1],$$

and since G is a basic encoder, $y^i\approx 0$ or $1/(1+D)$, $i=0,1,\cdots,k-1$. It follows from the invariant factor decomposition of F that the input sequences $[y^0,\cdots,y^{k-1}]$ such that $y^i\approx 0$ or $1/(1+D)$, and

$$[y^0,\cdots,y^{k-1}]G(I+\delta^2)\approx[0,\cdots,0],$$

form a linear space of dimension 1 or 2 according as $\Delta=1+D$ or $1+D^2$. It follows from Lemma 1 that either $x'=-x$, or $x'_{2j}=-x_{2j}$, $x'_{2j+1}=x_{2j+1}$, or $x'_{2j}=x_{2j}$, $x'_{2j+1}=-x_{2j+1}$. Therefore $(1-D^2)x/2\approx\mathbf{0}$ or $[(1-D^2)x/2]_{2j+s}=0$, $s=0$ or 1, for all but finitely many j. These are the only possible flawed output sequences. If $G^{(k)}=(g_1,\cdots,g_{k+1})$ then we have

Sequence c	Condition for c to be in the Sign Code
$\frac{1}{1+D}[1,1,\cdots,1]$	$\sum_{i=1}^{k+1} g_i\equiv 0 \bmod (1+D)$
$\frac{1}{1+D}[1,0,1,0,\cdots,1,0]$	$\sum_{i=1}^{(k+1)/2} g_{2i}\equiv 0\bmod(1+D)$
$\frac{1}{1+D}[0,1,0,1,\cdots,0,1]$	$\sum_{i=0}^{(k-1)/2} g_{2i+1}\equiv 0\bmod(1+D)$

In Example 4 the generator matrix for the sign code is $G=[1+D+D^2,1+D^2]$, and

$$\left[\frac{1}{1+D}\right]G\approx\frac{1}{1+D}[1,0].$$

The only flawed output sequences $(1-D^2)x/2$ satisfy $[(1-D^2)x/2]_{2j}=0$ for all but finitely many j.

If $\Delta=1+D$ and $\Sigma_{i=1}^{k+1}g_i\equiv 0\bmod(1+D)$, then there is a unique input sequence $[y^0,\cdots,y^{k-1}]$ such that $y^i\approx\mathbf{0}$ or $1/(1+D)$ and

$$[y^0,\cdots,y^{k-1}]G(I+\delta^2)\approx[\mathbf{0},\cdots,\mathbf{0}].$$

The input sequence $[y^0,\cdots,y^{k-1}]$ satisfies

$$[y^0,\cdots,y^{k-1}]G\approx\frac{1}{1+D}[1,1,\cdots,1].$$

The only flawed output sequence is the zero sequence. The problem of flaw correction reduces to that of limiting the zero-run length of possible output sequences, and we may apply Theorem 8.

Suppose for example that $\Sigma_{i=1}^{(k+1)/2}g_{2i}\equiv 0\bmod(1+D)$. Then there is an input sequence $[y^0,\cdots,y^{k-1}]$ such that $[y^0,\cdots,y^{k-1}]G\approx(1/(1+D))[1,0,\cdots,1,0]$. Output sequences $(1-D^2)x/2$ satisfying $[(1-D^2)x/2]_{2j}=0$ for all but finitely many j are flawed. Write $m^i_j=(-1)^{s^i_j}$, where $s^i_j=0$ or 1. If x is the codeword corresponding to messages m^i, $i=0,1,\cdots,k-1$, then $x_{jn+t}=(-1)^{z^t_j}$, where

$$[z^0,\cdots,z^k]=[s^0,\cdots,s^{k-1}]G+[a^0,\cdots,a^k].$$

The output sequence $(1-D^2)x/2$ is flawed if and only if $z^0\approx z^2\approx\cdots\approx z^k$. Let $c_0,c_1,\cdots,c_k$ be the columns of G and let $G_E=[c_0,c_2,\cdots,c_k]$, $G_O=[c_1,c_3,\cdots,c_{k-1}]$. If the columns of G_E are independent over $F[D]$, then the equation

$$[s^0,\cdots,s^{k-1}]G_E(I+\delta)=[a^0,\cdots,a^k](I+\delta)$$

always has a solution $[s^0,\cdots,s^{k-1}]$. It is therefore not possible to correct the flawed output sequences.

A similar argument applies to the case $\Sigma_{i=1}^{(k-1)/2}g_{2i+1}\equiv 0 \bmod (1+D)$. Note that since G is a basic encoder either the even columns are independent over $F[D]$ or the odd columns are independent (or both). Theorem 11 summarizes the above discussion.

Theorem 11: Let G be a noncatastrophic rate $k/(k+1)$ sign code for the $(1-D^2)/2$ channel, where k is odd. Let $F=G(I+\delta^2)$, and let Δ be the *gcd* of the $k\times k$ minors of F.

(1) The invariant factors of F are $1,1,\cdots,1,1+D$ or $1,1,\cdots,1,1+D,1+D$.

(2) If $\Delta=1+D$, there are two cases;

(a) $\Sigma_{i=1}^{k+1}g_i\equiv 0 \bmod (1+D)$; it is possible to limit the zero-run length using sequences $a^i(D)$ with period 2 and to eliminate flawed output sequences. It is not possible to accomplish this with a stationary trellis;

(b) $\Sigma_{i=1}^{(k+1)/2}g_{2i}\equiv 0 \bmod (1+D)$ and the columns of G_E are independent over $F[D]$, or $\Sigma_{i=1}^{(k-1)/2}g_{2i+1}\equiv 0 \bmod (1+D)$ and the columns of G_0 are independent over $F[D]$; it is not possible to eliminate flawed output sequences by an appropriate choice of coset. It is possible to limit the zero-run length using sequences $a^i(D)$ with period 2.

(3) If $\Delta=1+D^2$, then it is not possible to eliminate flawed output sequences by an appropriate choice of coset. It is possible to limit the zero-run length with a stationary trellis.

Finally we consider rate $k/(k+1)$ codes for the $(1-D^2)/2$ channel in the case where $k+1$ is odd. Here $s=1$ and $c=2$. By Lemma 6 the $1\times(k+1)$ matrix

$$\frac{F^{(k)}}{\Delta}=\frac{G^{(k)}(I+\delta^2+\delta^4+\cdots+\delta^{2(n-1)})^T}{\Delta}$$

is a basic syndrome former for the magnitude code F. By Lemmas 4 and 5 we have $\Delta=1,\ 1+D$, or $1+D^2$. If $G^{(k)}=(g_1,\cdots,g_{k+1})$, then by reducing the matrix $(I+\delta^2+\delta^4+\cdots+\delta^{2(n-1)})^T$ modulo $1+D^2$, it follows that

$$\Delta=1 \text{ if and only if } \sum_{i=1}^{k+1}g_i\equiv 1 \bmod (1+D). \qquad (2)$$

If $\Delta\neq 1$, then

$$\Delta=1+D^2 \text{ if and only if } G^{(k)}(1,D,1,D,\cdots,1)^T \equiv 0 \bmod (1+D^2). \qquad (3)$$

(Compare (2) and (3) with the hypotheses of Theorem 1 of Baumert, McEliece, and van Tilborg [18].) If $\Delta=1+D$ and there exist $\mathbb{F}_2$-messages $y^i(D)=0$ or $1/(1+D)$, $i=0,1,\cdots,k-1$, such that

$$[y^0(D),\cdots,y^{k-1}(D)]G(I+\delta^2)\approx[0,\cdots,0],$$

then since $n=k+1$ is odd,

$$[y^0(D),\cdots,y^{k-1}(D)]G\approx\frac{1}{1+D}[1,\cdots,1].$$

In this case the problem of flaw correction reduces to that of limiting the zero-run length of possible output sequences. (Note that if $\Delta=1+D^2$, then the invariant factors of F are $1,1,\cdots,1,1+D^2$.) If $\Delta=1+D^2$, then an argument similar to that given in Theorem 11 shows that if the columns of G are independent over $F[D]$, then it is impossible to eliminate flawed output sequences by an appropriate choice of coset. We have proved the following theorem.

Theorem 12: Let G be a noncatastrophic rate $k/(k+1)$ sign code for the $(1-D^2)/2$ channel, where k is even. Let $F=G(I+\delta^2)$ and let Δ be the *gcd* of the $k\times k$ minors of F. Suppose that the columns of G are independent over $F[D]$.

1) $\Delta=1,\ 1+D$, or $1+D^2$. Further

$$\Delta=1 \text{ if and only if } G^{(k)}(1,1,\cdots,1)^T\equiv 1 \bmod (1+D)$$

and if $\Delta\neq 1$, then

$$\Delta=1+D^2 \text{ if and only if } G^{(k)}(1,D,1,D,\cdots,1) \equiv 0 \bmod (1+D^2).$$

2) If $\Delta=1$, then the recording code is not flawed, but limiting the zero-run length requires sequences $a^i(D)$ with period greater than 2.

3) If $\Delta=1+D$, then it is possible to eliminate flawed output sequences and to limit the zero-run.

4) If $\Delta=1+D^2$, then it is not possible to eliminate flawed output sequences by an appropriate choice of coset. It is possible to limit the zero-run length using a stationary trellis.

We conclude with an example in which one column of the magnitude code F is identically zero.

Example: A rate 2/3 code for the $(1-D^2)/2$ channel. The sign code G and the magnitude code F are given by

$$G=\begin{bmatrix}D&1&1\\D&1&0\end{bmatrix} \text{ and } F=\begin{bmatrix}0&1+D&1+D\\0&1&D\end{bmatrix}.$$

The 2×2 minors of F are $f_1=1+D^2$, $f_2=f_3=0$, so that $\Delta=1+D^2$. Messages $y^0(D)$, $y^1(D)$ for which

$$[y^0(D),\ y^1(D)]F\approx[0,0,0]$$

are $[y^0(D),\ y^1(D)]=[1/(1+D^2),1/(1+D)],[1/(1+D),0],[D/(1+D^2),1/(1+D)]$. If we modify the encoding rules by taking

$$x_{3k}=(-1)^k m_{k-1}^0 m_{k-1}^1,$$

$$x_{3k+1}=(-1)^{k+1}m_k^0 m_k^1,\ x_{3k+2}=(-1)^k m_k^0$$

so that

$$s_{3k}=(-1)^k m_{k-1}^0 m_{k-1}^1,$$

$$s_{3k+1}=\frac{1}{2}\left[(-1)^{k+1}m_k^0m_k^1-(-1)^{k-1}m_{k-1}^0\right],$$

$$s_{3k+2}=\frac{1}{2}\left[(-1)^k m_k^0-(-1)^k m_{k-1}^0 m_{k-1}^1\right],$$

then the modified code is no longer flawed and the minimum squared distance is 4. We leave it as an exercise for the reader to prove that if F is a generator matrix for a magnitude code and one column of F is identically zero, then the minimum squared distance cannot exceed 4.

VII. Tables of Codes

Generator polynomials for the codes listed below are given in octal form; the octal representation of $D^4 + D^3 + D^2 + 1$ is 35. Codes with flaws that cannot be corrected by choosing an appropriate coset of the sign code to generate inputs are marked with a star. All flawed codes have the property that the fraction of messages of length M corresponding to flawed output sequences tends to O as M tends to ∞. (In Example 4 there are essentially four pairs of bad messages.) In practice it may be quite complicated to constrain the input to the encoder to avoid these bad messages. However, it is theoretically possible to do this at some marginal loss in data rate. We include

TABLE II
RATE 1 / 2 CODES FOR THE $(1-D)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1,1	3,0	$1+D$	6	1	$\frac{1}{1+D}[1,0]$	1
1,2	5,3	$1+D$	4	2	$\frac{1}{1+D}[1,0]$	3
13,2	17,11	$1+D$	6	3	$\frac{1}{1+D}[1,0]$	6
1,26	55,27	$1+D$	8	5	$\frac{1}{1+D}[1,0]$	9
153,16	167,145	$1+D$	10	6	$\frac{1}{1+D}[1,0]$	12
165,314	755,271	$1+D$	12	8	$\frac{1}{1+D^2}[0,1]$	17

TABLE III
RATE 1 / 2 CODES FOR THE $(1-D^2)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1,1	3,3	$1+D$	4	1	$\frac{1}{1+D^2}[1,0]$	2
5,7*	17,11	$1+D$	6	3	$\frac{1}{1+D^2}[1,0]$	6
13,15	35,27	$1+D$	8	4	$\frac{1}{1+D^2}[1,0]$	8
43,65*	145,137	$1+D$	10	6	$\frac{1}{1+D^2}[0,1]$	12
205,323	617,565	$1+D$	12	8	$\frac{1}{1+D^2}[1,0]$	16

TABLE IV
$R=1/2$ CODES FOR THE $(1-D^3)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1,0	1,2	1	2	1	$\frac{1}{1+D^2}[0,1]$	4
1,1	5,3	$1+D$	4	2	$\frac{1}{1+D}[1,0]$	3
7,3	13,15	1	6	3	$\frac{1}{1+D^2}[0,1]$	8
15,1	11,33	$1+D$	6	4	$\frac{1}{1+D}[0,1]$	4
23,13*	77,55	$1+D^3$	8	5	$\frac{1}{1+D}[1,0]$	6
67,31*	123,167	$1+D^3$	10	6	$\frac{1}{1+D}[1,0]$	8
221,121*	725,563	$1+D^3$	12	8	$\frac{1}{1+D}[1,0]$	12

TABLE V
$R = 2/3$ CODES FOR THE $(1 - D)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
3 1 0 2 1 2	3 2 1 6 3 3	$1 + D$	4	3	$\frac{1}{1+D}[1,0,0]$	8
3 7 0 2 4 7	3 4 7 14 6 3	$1 + D$	6	5	$\frac{1}{1+D}[1,0,0]$	14

TABLE VI
RATE $2/3$ CODES FOR THE $(1 - D^2)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1 1 1* 0 1 3	3 3 0 2 7 3	$1 + D^2$	4	3	$\frac{1}{1+D}[1,0,0]$	5
3 2 2* 2 3 5	7 6 1 4 11 7	$1 + D^2$	6	5	$\frac{1}{1+D}[0,0,1]$	11
1 1 3* 4 7 7	3 7 2 12 11 3	$1 + D^2$	6	5	$\frac{1}{1+D}[1,0,0]$	11
7 6 4 2 7 11	13 17 3 14 20 13	1	8	7	$\frac{1}{1+D^3}[0,1,0]$	25

TABLE VII
RATE $2/3$ FOR THE $(1 - D^3)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1 1 0* 0 1 1	3 3 0 0 3 3	$1 + D^2$	4	2	$\frac{1}{1+D^2}[1,0,0]$	4
3 3 2* 2 4 5	5 5 6 6 14 17	$1 + D^2$	6	5	$\frac{1}{1+D^2}[1,0,0]$	11
1 10 13* 4 3 5	3 30 35 14 5 17	$1 + D^2$	8	7	$\frac{1}{1+D^2}[0,1,0]$	19

TABLE VIII
RATE $3/4$ CODES FOR THE $(1 - D)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1 0 0 1 0 1 2 0 2 0 1 0	3 1 0 1 0 1 3 2 2 2 1 1	$1 + D$	4	3	$\frac{1}{1+D}[0,0,0,1]$	12

TABLE IX
RATE 3 / 4 CODES FOR THE $(1 - D^2)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1 1 0 0*	1 1 1 1					
0 1 0 2	0 5 0 3	$1 + D^2$	4	3	$\frac{1}{1+D}[0,0,0,1]$	8
0 0 1 1	2 2 1 1					

TABLE X
RATE 3 / 4 CODES FOR THE $(1 - D^3)/2$ CHANNEL

Sign Code G	Magnitude Code F	Δ	d^2	Memory	Coset $a(D)$	Zero-Run Length
1 1 0 0	1 1 1 1					
0 1 0 2	0 5 0 3	$1 + D^2$	4	3	$\frac{1}{1+D}[1,0,0,0]$	8
0 0 1 1	2 2 1 1					

codes for the $1 - D^3$ channel to illustrate Corollary 9 and to further illustrate the fact that flawed codes have good zero-run length properties.

VIII. CONCLUSION

Motivated by an idealized model of the magnetic recording channel, we have designed codes for a partial response channel with transfer function $(1 - D^N)/2$. Channel inputs are generated using a nontrivial coset of a binary convolutional code called the sign code. The probability of decoder error is determined by the minimum squared Euclidean distance between outputs corresponding to distinct inputs. This Euclidean distance is bounded below by the free distance of a second binary convolutional code called the magnitude code. The coset of the sign code is chosen to limit the zero-run length of the output of the channel and we have shown how to select an appropriate coset. We have analyzed the performance of rate $k/(k + 1)$ codes on the $(1 - D)/2$ and $(1 - D^2)/2$ channels. Recording codes for which the magnitude code admits nontrivial invariant factors (that is, catastrophic magnitude codes) can outperform magnitude codes with trivial invariant factors.

One problem demanding further study is the design of trellis codes for partial response channels with more complicated transfer functions. We note that transfer functions arising in optical-magnetic recording need not involve $(1 - D)$ as a factor. A first step would be to design codes for a transfer function $f(D) = (D^{N_1} - l_1) \cdots (D^{N_s} - l_s)$ where N_i, l_i are positive integers.

A second problem is the design of codes with spectral nulls at certain frequencies. This is important when writing data on disks with an embedded servo system. There are fixed frequencies f_1, f_2 and the servo signal e is the amplitude of f_2 minus the amplitude of f_1. If $e > 0$ then the head is moved left and if $e < 0$ then the head is moved right. Recent work by Marcus and Siegel [19] and by Ancheta, Hassner, and Howell [20] concerns the encoding of input data as run-length limited sequences with spectral nulls at certain frequencies using finite state machines. These authors consider run-length limited sequences because they are trying to minimize intersymbol interference. An alternative approach is to combine a finite state machine with a code designed to exploit intersymbol interference. Since we are no longer concerned with run-length constraints the finite state machine may well be less complicated.

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